

IMPROVED UNIQUENESS CONDITIONS FOR CANONICAL
TENSOR DECOMPOSITIONS WITH LINEARLY DEPENDENT
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Abstract. In this paper, we derive improved uniqueness conditions for a constrained version of the canonical order-3 tensor decomposition, also known as Candecomp/Parafac (CP). CP decomposes a three-way array into a prespecified number of outer product arrays. The constraint is that some vectors forming the outer product arrays are linearly dependent according to a prespecified pattern. This is known as the PARALIND family of decompositions. We provide both uniqueness conditions and partial uniqueness conditions for PARALIND, and show that these are improved and more precise variants of existing conditions. Our results are illustrated by means of examples.

Key words. PARALIND, CONFAC, parafac, candecomp, tensor decomposition, uniqueness

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1. Introduction. Tensors of order 3 are defined on the outer product of three linear spaces, \mathcal{T}_ℓ , $\ell = 1, 2, 3$. Once bases of spaces \mathcal{T}_ℓ are fixed, they can be represented by three-way arrays. For simplicity, tensors are usually assimilated with their array representation.

The canonical order-3 tensor decomposition is of the form

$$(1.1) \quad \underline{\mathbf{X}} = \sum_{r=1}^R (\mathbf{a}_r \circ \mathbf{b}_r \circ \mathbf{c}_r),$$

where $\underline{\mathbf{X}}$ is an $I \times J \times K$ tensor (or array), \mathbf{a}_r ($I \times 1$), \mathbf{b}_r ($J \times 1$), and \mathbf{c}_r ($K \times 1$) are vectors, and \circ denotes the outer vector product. For vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$, the outer vector product $\mathbf{a} \circ \mathbf{b} \circ \mathbf{c}$ is an order-3 tensor with entries $a_i b_j c_k$. We refer to $\underline{\mathbf{X}}$ in (1.1) as having three *modes*. The entry x_{ijk} of $\underline{\mathbf{X}}$ is the entry in row i , column j , and frontal slice k . Let $\mathbf{A} = [\mathbf{a}_1 \dots \mathbf{a}_R]$, $\mathbf{B} = [\mathbf{b}_1 \dots \mathbf{b}_R]$, and $\mathbf{C} = [\mathbf{c}_1 \dots \mathbf{c}_R]$ denote the component matrices. We denote an order-3 decomposition (1.1) as $(\mathbf{A}, \mathbf{B}, \mathbf{C})$. Note that when the modes of $\underline{\mathbf{X}}$ are permuted in (1.1), the component matrices are permuted identically.

An order-3 tensor has rank 1 if it can be written as the outer product of three vectors. The rank of an order-3 tensor $\underline{\mathbf{X}}$ is defined as the smallest number of rank-1 tensors whose sum equals $\underline{\mathbf{X}}$. Hence, (1.1) decomposes $\underline{\mathbf{X}}$ into R rank-1 terms. Hitchcock [17], [18] introduced tensor rank and the related tensor decomposition (1.1), also for order $n \geq 3$. The same decomposition was proposed independently by Carroll and Chang [5] and Harshman [16] for component analysis of tensors. They named it

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Candecomp and Parafac, respectively, and we refer to (1.1) as the Candecomp/Parafac (CP) decomposition.

For a given order-3 tensor and number R of rank-1 components, a best-fitting decomposition (1.1) is usually found by an iterative algorithm. The most well-known algorithm is alternating least squares. A comparison of algorithms can be found in Tomasi and Bro [43]. Note that a best-fitting decomposition is a best rank- R approximation of the tensor.

For later use, we mention that the CP decomposition (1.1) is a special case of the Tucker3 decomposition [44]. The latter is defined as

$$(1.2) \quad \underline{\mathbf{X}} = \sum_{r=1}^R \sum_{p=1}^P \sum_{q=1}^Q g_{rpq} (\mathbf{a}_r \circ \mathbf{b}_p \circ \mathbf{c}_q).$$

Clearly, the case with $R = P = Q$ and $g_{rpq} = 0$ if $(r, p, q) \neq (r, r, r)$ yields (1.1). The $R \times P \times Q$ array $\underline{\mathbf{G}}$ with entries g_{rpq} is referred to as the core array. In Tucker3, the matrices $\mathbf{A} = [\mathbf{a}_1 \dots \mathbf{a}_R]$, $\mathbf{B} = [\mathbf{b}_1 \dots \mathbf{b}_P]$, and $\mathbf{C} = [\mathbf{c}_1 \dots \mathbf{c}_Q]$ are the component matrices.

CP and Tucker3 can be seen as generalizations of principal component analysis for matrices. They can be used for exploratory component analysis of three-way data. Such (real-valued) applications of CP and Tucker3 occur in psychology [24] [21] and chemometrics [33]. Complex-valued CP is used in, e.g., signal processing and telecommunications research [30], [31], [12]. Here, the decompositions are mostly used to separate signal sources from an observed mixture of signals. CP of order 4 describes the basic structure of fourth-order cumulants of multivariate data on which many algebraic methods for independent component analysis (ICA) are based [6], [8], [11], [7]. A general overview of applications of CP and Tucker3 can be found in [22], [1].

A potential problem of computing a best-fitting CP decomposition (1.1) is that an optimal solution may not exist. Indeed, a tensor may not have a best rank- R approximation. This is due to the fact that the set of tensors of rank at most R is not closed for $R \geq 2$; see [14]. In such cases, diverging components (i.e., close to linear dependence and large in magnitude) occur while running an iterative algorithm designed to find a best rank- R approximation; see [26], [23], [34], [35]. This problem can be fixed by including interaction terms in the CP decomposition [40], [28], [38].

An attractive feature of the CP decomposition (1.1) is that the component matrices are unique up to a simultaneous column permutation and columnwise rescaling under relatively mild conditions. Formally we define uniqueness up to permutation and scaling of $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ as follows.

DEFINITION 1.1. *The CP decomposition $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ is called unique up to permutation and scaling if any alternative decomposition $(\bar{\mathbf{A}}, \bar{\mathbf{B}}, \bar{\mathbf{C}})$ satisfies $\bar{\mathbf{A}} = \mathbf{A} \boldsymbol{\Pi} \boldsymbol{\Lambda}_a$, $\bar{\mathbf{B}} = \mathbf{B} \boldsymbol{\Pi} \boldsymbol{\Lambda}_b$, and $\bar{\mathbf{C}} = \mathbf{C} \boldsymbol{\Pi} \boldsymbol{\Lambda}_c$, with $\boldsymbol{\Pi}$ an $R \times R$ permutation matrix, and $\boldsymbol{\Lambda}_a, \boldsymbol{\Lambda}_b, \boldsymbol{\Lambda}_c$ nonsingular diagonal matrices such that $\boldsymbol{\Lambda}_a \boldsymbol{\Lambda}_b \boldsymbol{\Lambda}_c = \mathbf{I}_R$.*

Hence, a CP decomposition (1.1) is unique up to permutation and scaling if the only ambiguities it contains are the permutation of the R rank-1 components, and the scaling of the three vectors constituting each rank-1 component.

The classical uniqueness condition for CP is due to Kruskal [25]. Kruskal's condition relies on a particular concept of matrix rank that he introduced, which has been named k-rank after him. Specifically, the k-rank of a matrix is the largest number x such that every subset of x columns of the matrix is linearly independent. We denote

the k-rank of a matrix \mathbf{A} as $k_{\mathbf{A}}$. For a CP decomposition $(\mathbf{A}, \mathbf{B}, \mathbf{C})$, Kruskal [25] proved that

$$(1.3) \quad 2R + 2 \leq k_{\mathbf{A}} + k_{\mathbf{B}} + k_{\mathbf{C}}$$

is a sufficient condition for uniqueness up to permutation and scaling. A more condensed and accessible proof of (1.3) was given by Stegeman and Sidiropoulos [39]. See Rhodes [27] for a different approach. Kruskal's uniqueness condition was generalized to order $n \geq 3$ by Sidiropoulos and Bro [29].

The (mode-3) matrix unfolding of the CP decomposition $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ is given by

$$(1.4) \quad (\mathbf{A} \odot \mathbf{B}) \mathbf{C}^T,$$

where \odot denotes the (columnwise) Khatri–Rao product: $(\mathbf{A} \odot \mathbf{B}) = [\mathbf{a}_1 \otimes \mathbf{b}_1 \dots \mathbf{a}_R \otimes \mathbf{b}_R]$, with \otimes denoting the Kronecker product.

Less restrictive uniqueness conditions than (1.3) have been obtained for the case where (at least) one of the component matrices has full column rank; i.e., the vectors in (at least) one mode j are linearly independent. In this case, the uniqueness condition does not depend on component matrix j . Moreover, alternative decompositions are found only in the column space of the Khatri–Rao product of all component matrices except the j th one. See Jiang and Sidiropoulos [19] (order 3), De Lathauwer [9] (order 3 and 4), and Stegeman [36] (order 3). See Stegeman [37] for a generalization to order $n \geq 3$.

In this paper, we consider uniqueness conditions for order-3 CP with linear dependencies in the columns of the component matrices $\mathbf{A}, \mathbf{B}, \mathbf{C}$. In particular, the patterns of the linear dependencies are known and fixed. This type of decomposition is introduced in Bro et al. [4], and is named PARALIND (parallel profiles with linear dependencies). Instead of $(\mathbf{A}, \mathbf{B}, \mathbf{C})$, a PARALIND decomposition is given by $(\tilde{\mathbf{A}}\Psi, \tilde{\mathbf{B}}\Phi, \tilde{\mathbf{C}}\Omega)$, where $\tilde{\mathbf{A}}$ is $I \times R_1$, $\tilde{\mathbf{B}}$ is $J \times R_2$, $\tilde{\mathbf{C}}$ is $K \times R_3$, Ψ is $R_1 \times R$, Φ is $R_2 \times R$, and Ω is $R_3 \times R$. The matrices Ψ , Φ , and Ω are fixed and contain the patterns of linear dependency of the columns of $\mathbf{A}\Psi$, $\mathbf{B}\Phi$, and $\mathbf{C}\Omega$, respectively. We refer to Ψ , Φ , and Ω as the constraint matrices.

Throughout, we assume $\mathbf{A}, \mathbf{B}, \mathbf{C}$ to have full column rank, and Ψ, Φ, Ω to have full row rank. In other cases, the PARALIND decomposition has a superfluous factor. Indeed, suppose $\text{rank}(\tilde{\mathbf{A}}) < R_1$, and $\tilde{\mathbf{A}}\mathbf{d} = \mathbf{0}$. Then $\tilde{\mathbf{A}}\Psi = (\tilde{\mathbf{A}}\mathbf{D})(\mathbf{D}^{-1}\Psi)$, with \mathbf{D} nonsingular such that $\tilde{\mathbf{A}}\mathbf{D} = [\tilde{\mathbf{a}}_1 \dots \tilde{\mathbf{a}}_{p-1} \mathbf{0} \tilde{\mathbf{a}}_{p+1} \dots \tilde{\mathbf{a}}_{R_1}]$ for some column p (vector \mathbf{d} is the p th column of \mathbf{D}). Hence, the PARALIND decomposition is equivalent to a PARALIND decomposition in which $\tilde{\mathbf{A}}$ is replaced by $[\tilde{\mathbf{a}}_1 \dots \tilde{\mathbf{a}}_{p-1} \tilde{\mathbf{a}}_{p+1} \dots \tilde{\mathbf{a}}_{R_1}]$, and Ψ is replaced by $\mathbf{D}^{-1}\Psi$ with its p th row deleted. It can be shown analogously that the PARALIND decomposition has a superfluous factor if one of Ψ, Φ, Ω does not have full row rank. We also assume that Ψ, Φ, Ω do not contain all-zero columns, which guarantees R rank-1 terms in the PARALIND decomposition.

In Bro et al. [4], PARALIND decompositions are used to analyze flow injection data and fluorescence data. In de Almeida, Favier, and Mota [2], [3] PARALIND decompositions are used to model multiple-antenna transmissions in the context of wireless telecommunications and signal processing. Related works in signal processing are [32], [13]. In [3], the constraint matrices Ψ, Φ, Ω have columns from the $R_j \times R_j$ identity matrix, $j = 1, 2, 3$. This form of PARALIND is called CONFAC (constrained factors) by [3].

For a given data array and constraint matrices Ψ, Φ, Ω , the best-fitting PARALIND component matrices $\tilde{\mathbf{A}}, \tilde{\mathbf{B}}, \tilde{\mathbf{C}}$ can be found using a similar alternating least

squares algorithm as can be used for the CP decomposition; see [4] and [3]. As for CP, there may not be an optimal solution for a particular PARALIND decomposition.

As an example of PARALIND, let $R_1 = R_2 = R_3 = 3$, $R = 4$, and

$$(1.5) \quad \Psi = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad \Phi = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad \Omega = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

We have $\tilde{\mathbf{A}}\Psi = [\tilde{\mathbf{a}}_1 \ \tilde{\mathbf{a}}_2 \ \tilde{\mathbf{a}}_3 \ \tilde{\mathbf{a}}_1]$, $\tilde{\mathbf{B}}\Phi = [\tilde{\mathbf{b}}_1 \ \tilde{\mathbf{b}}_2 \ \tilde{\mathbf{b}}_3 \ \tilde{\mathbf{b}}_2]$, and $\tilde{\mathbf{C}}\Omega = [\tilde{\mathbf{c}}_1 \ \tilde{\mathbf{c}}_2 \ \tilde{\mathbf{c}}_3 \ \tilde{\mathbf{c}}_3]$. The rank-1 terms of the PARALIND decomposition are given by

$$(1.6) \quad (\tilde{\mathbf{a}}_1 \circ \tilde{\mathbf{b}}_1 \circ \tilde{\mathbf{c}}_1) + (\tilde{\mathbf{a}}_2 \circ \tilde{\mathbf{b}}_2 \circ \tilde{\mathbf{c}}_2) + (\tilde{\mathbf{a}}_3 \circ \tilde{\mathbf{b}}_3 \circ \tilde{\mathbf{c}}_3) + (\tilde{\mathbf{a}}_1 \circ \tilde{\mathbf{b}}_2 \circ \tilde{\mathbf{c}}_3).$$

As shown by [3], a PARALIND decomposition can be written as a Tucker3 decomposition with a constrained $R_1 \times R_2 \times R_3$ core array

$$(1.7) \quad \underline{\mathbf{G}} = \sum_{r=1}^R (\psi_r \circ \phi_r \circ \omega_r),$$

where ψ_r , ϕ_r , and ω_r are the r th columns of Ψ , Φ , and Ω , respectively. Hence, the core array $\underline{\mathbf{G}}$ satisfies a CP decomposition with component matrices Ψ , Φ , Ω . Constrained Tucker3 models have applications in chemometrics; see [33]. In the example (1.5)–(1.6), the Tucker3 core is $3 \times 3 \times 3$ with frontal slices

$$(1.8) \quad \left[\begin{array}{ccc|ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right].$$

A systematic treatment of uniqueness properties of the PARALIND component matrices $\tilde{\mathbf{A}}$, $\tilde{\mathbf{B}}$, $\tilde{\mathbf{C}}$ is presented in Stegeman and de Almeida [41]. The uniqueness property is considered for each component matrix separately. A distinction is made between uniqueness up to permutation and scaling and partial uniqueness. For CP, the term partial uniqueness has been used to describe cases where some columns of a component matrix are identified up to their linear span only, or where only a finite number of alternative CP solutions are available (up to permutation and scaling); see Ten Berge [42]. For PARALIND, we adopt the first definition, and we call $\tilde{\mathbf{A}}$ partially unique if its columns can be partitioned into disjoint subsets and each subset is identified up to its linear span. The same definition is used by Bro et al. [4] and by [41]. Results on partial uniqueness for CP or PARALIND have been mostly ad hoc. The only systematic approach is found in [41], and in this paper. The analysis of [41] is based on the approach of Jiang and Sidiropoulos [19] for CP with full column rank in (at least) one mode. Below, we show that the idea of Guo et al. [15] to study CP uniqueness by splitting up a CP decomposition into smaller decompositions when one component matrix is unique up to permutation and scaling can also be used to obtain partial uniqueness results for PARALIND.

In this paper, we present improved and more precise variants of the main PARALIND uniqueness conditions of [41]. We reprove the uniqueness conditions of [41] using simpler proofs, and show that our improved uniqueness conditions follow naturally from these proofs. The reason why the proofs are simpler is because we do not use Kruskal's [25] permutation lemma (in case of uniqueness up to permutation and

scaling) or De Lathauwer's [10] equivalence lemma for partitioned matrices (in case of partial uniqueness). Our uniqueness conditions are relatively easy to check with the use of (symbolic) linear algebra computation software, and the improvement with respect to the results of [41] is demonstrated by means of examples.

A different approach to obtain conditions for PARALIND uniqueness (up to permutation and scaling) is via the Kruskal-type uniqueness results of [15] for CP with proportional vectors in one mode. We also compare our uniqueness conditions with those obtained via the approach of [15] for CP. Our examples show that it depends on the particular PARALIND decomposition which type of uniqueness condition is more powerful or appropriate.

In applications of PARALIND decompositions, the component matrices can be real-valued [4] or complex-valued [2], [3]. To the best of our knowledge, in all PARALIND applications the constraint matrices are real-valued. Our results and those in [41] are proven for real component matrices. However, they can be translated easily to the complex case. We will elaborate on this in the discussion section at the end of this paper.

This paper is organized as follows. Section 2 states conditions for uniqueness up to permutation and scaling of one PARALIND component matrix, and includes our improved condition. Section 3 explains how our PARALIND uniqueness condition can be checked more easily with the help of (symbolic) linear algebra computation software. Section 4 contains examples in which we apply the uniqueness conditions of section 2. In section 5 we present partial uniqueness conditions for one PARALIND component matrix, including our improved condition. These conditions are applied in the examples contained in section 6. Finally, section 7 contains a discussion of our findings.

2. Uniqueness conditions for PARALIND. Here, we present conditions for uniqueness up to permutation and scaling of one PARALIND component matrix. Due to the linear dependencies in the columns of $\tilde{\mathbf{A}}\Psi$, $\tilde{\mathbf{B}}\Phi$, and $\tilde{\mathbf{C}}\Omega$, there is less freedom of scaling/counterscaling in the vectors constituting each rank-1 term than there is in CP. Also, jointly permuting the order of the columns of $\tilde{\mathbf{A}}$, $\tilde{\mathbf{B}}$, $\tilde{\mathbf{C}}$ may result in a PARALIND decomposition with different constraint matrices. To avoid these complications, uniqueness in PARALIND is considered for each component matrix separately. See [41, section 3] for more details. We define uniqueness up to permutation and scaling of a PARALIND component matrix as follows.

DEFINITION 2.1. *Let the PARALIND decomposition $(\tilde{\mathbf{A}}\Psi, \tilde{\mathbf{B}}\Phi, \tilde{\mathbf{C}}\Omega)$ be such that $\tilde{\mathbf{A}}, \tilde{\mathbf{B}}, \tilde{\mathbf{C}}$ have full column rank, and Ψ, Φ, Ω have full row rank and no all-zero columns. The component matrix $\tilde{\mathbf{A}}$ is called unique up to permutation and scaling if any alternative decomposition $(\tilde{\mathbf{A}}\Psi, \tilde{\mathbf{B}}\Phi, \tilde{\mathbf{C}}\Omega)$ satisfies $\tilde{\mathbf{A}} = \tilde{\mathbf{A}}\Pi\Lambda_a$, with Π an $R_1 \times R_1$ permutation matrix, and Λ_a a nonsingular diagonal matrix.*

In section 2.1, we state the uniqueness condition for one PARALIND component matrix of Stegeman and de Almeida [41] and our improved version of it. In section 2.2, we state a PARALIND uniqueness condition derived from a CP uniqueness condition of Guo et al. [15].

2.1. Uniqueness conditions for one PARALIND component matrix. Before we present the PARALIND uniqueness result of Stegeman and de Almeida [41], we define three matrix unfoldings of a PARALIND decomposition. Let

$$(2.1) \quad \mathbf{G}_1 = (\Phi \odot \Omega) \Psi^T, \quad \mathbf{G}_2 = (\Omega \odot \Psi) \Phi^T, \quad \mathbf{G}_3 = (\Psi \odot \Phi) \Omega^T.$$

The mode-1 matrix unfolding of the PARALIND decomposition $(\tilde{\mathbf{A}}\Psi, \tilde{\mathbf{B}}\Phi, \tilde{\mathbf{C}}\Omega)$ is defined as

$$(2.2) \quad (\tilde{\mathbf{B}}\Phi \odot \tilde{\mathbf{C}}\Omega)(\tilde{\mathbf{A}}\Psi)^T = (\tilde{\mathbf{B}} \otimes \tilde{\mathbf{C}})\mathbf{G}_1 \tilde{\mathbf{A}}^T.$$

Its mode-2 matrix unfolding is defined as

$$(2.3) \quad (\tilde{\mathbf{C}}\Omega \odot \tilde{\mathbf{A}}\Psi)(\tilde{\mathbf{B}}\Phi)^T = (\tilde{\mathbf{C}} \otimes \tilde{\mathbf{A}})\mathbf{G}_2 \tilde{\mathbf{B}}^T,$$

and its mode-3 matrix unfolding is defined as

$$(2.4) \quad (\tilde{\mathbf{A}}\Psi \odot \tilde{\mathbf{B}}\Phi)(\tilde{\mathbf{C}}\Omega)^T = (\tilde{\mathbf{A}} \otimes \tilde{\mathbf{B}})\mathbf{G}_3 \tilde{\mathbf{C}}^T.$$

Stegeman and de Almeida [41] prove the following PARALIND uniqueness condition. Let

$$(2.5) \quad N_j^{(1)} = \text{rank}(\Phi \text{diag}(\psi_j^T) \Omega^T), \quad j = 1, \dots, R_1,$$

$$(2.6) \quad N_j^{(2)} = \text{rank}(\Omega \text{diag}(\phi_j^T) \Psi^T), \quad j = 1, \dots, R_2,$$

$$(2.7) \quad N_j^{(3)} = \text{rank}(\Psi \text{diag}(\omega_j^T) \Phi^T), \quad j = 1, \dots, R_3,$$

where ψ_j^T denotes row j of Ψ , ϕ_j^T denotes row j of Φ , and ω_j^T denotes row j of Ω . Let $w(\cdot)$ denote the number of nonzero entries of a vector.

THEOREM 2.2. *Let the PARALIND decomposition $(\tilde{\mathbf{A}}\Psi, \tilde{\mathbf{B}}\Phi, \tilde{\mathbf{C}}\Omega)$ be such that $\tilde{\mathbf{A}}, \tilde{\mathbf{B}}, \tilde{\mathbf{C}}$ have full column rank, and Ψ, Φ, Ω have full row rank and no all-zero columns. The component matrix $\tilde{\mathbf{A}}$ is unique up to permutation and scaling if \mathbf{G}_1 has full column rank, and if for any nonzero vector \mathbf{d} ,*

$$(2.8) \quad \text{rank}(\Phi \text{diag}(\Psi^T \mathbf{d}) \Omega^T) \leq \max(N_1^{(1)}, \dots, N_{R_1}^{(1)}) \quad \text{implies} \quad w(\mathbf{d}) = 1.$$

Proof. See Stegeman and de Almeida [41, section 4.1]. See Appendix A for a simpler proof. \square

For uniqueness of $\tilde{\mathbf{A}}$ it is necessary that \mathbf{G}_1 has full column rank; see [41, section 4.2]. Our improved version of Theorem 2.2 is as follows.

THEOREM 2.3. *Let the PARALIND decomposition $(\tilde{\mathbf{A}}\Psi, \tilde{\mathbf{B}}\Phi, \tilde{\mathbf{C}}\Omega)$ be such that $\tilde{\mathbf{A}}, \tilde{\mathbf{B}}, \tilde{\mathbf{C}}$ have full column rank, and Ψ, Φ, Ω have full row rank and no all-zero columns. The component matrix $\tilde{\mathbf{A}}$ is unique up to permutation and scaling if $\mathbf{G}_1, \mathbf{G}_2, \mathbf{G}_3$ have full column rank, and if for any set of R_1 linearly independent vectors $\mathbf{d}_1, \dots, \mathbf{d}_{R_1}$,*

$$(2.9) \quad \begin{aligned} \text{rank}(\Phi \text{diag}(\Psi^T \mathbf{d}_j) \Omega^T) &= N_j^{(1)}, \quad j = 1, \dots, R_1, \quad \text{implies} \\ w(\mathbf{d}_j) &= 1, \quad j = 1, \dots, R_1. \end{aligned}$$

Proof. See Appendix A for the proof. \square

The proofs of Theorems 2.2 and 2.3 are in Appendix A. Our proof of Theorem 2.2 is simpler than the one given in [41], since we do not use Kruskal's [25] permutation lemma.

It can be seen that condition (2.8) implies condition (2.9). Hence, if not only \mathbf{G}_1 but also \mathbf{G}_2 and \mathbf{G}_3 have full column rank, then Theorem 2.3 is a relaxation of Theorem 2.2. As we will see in section 4, cases for which condition (2.9) holds but not condition (2.8) are easy to find in the literature.

By interchanging the roles of $\tilde{\mathbf{A}}\Psi$, $\tilde{\mathbf{B}}\Phi$, and $\tilde{\mathbf{C}}\Omega$ in Theorems 2.2 and 2.3, conditions for uniqueness of $\tilde{\mathbf{B}}$ and $\tilde{\mathbf{C}}$ can be obtained. The roles of $\mathbf{G}_1, \mathbf{G}_2, \mathbf{G}_3$ and $N_j^{(1)}, N_j^{(2)}, N_j^{(3)}$ must be interchanged correspondingly.

For later use, we state the following result.

PROPOSITION 2.4. *Let the PARALIND decomposition $(\tilde{\mathbf{A}}\Psi, \tilde{\mathbf{B}}\Phi, \tilde{\mathbf{C}}\Omega)$ be such that $\tilde{\mathbf{A}}, \tilde{\mathbf{B}}, \tilde{\mathbf{C}}$ have full column rank, and Ψ, Φ, Ω have full row rank and no all-zero columns. Moreover, let the PARALIND decomposition have alternative $(\bar{\mathbf{A}}\Psi, \bar{\mathbf{B}}\Phi, \bar{\mathbf{C}}\Omega)$ with the same residuals.*

- (i) *If \mathbf{G}_1 has full column rank, then $\bar{\mathbf{A}} = \tilde{\mathbf{A}}\mathbf{S}$ for some nonsingular matrix \mathbf{S} .*
- (ii) *If \mathbf{G}_2 has full column rank, then $\bar{\mathbf{B}} = \tilde{\mathbf{B}}\mathbf{T}$ for some nonsingular matrix \mathbf{T} .*
- (iii) *If \mathbf{G}_3 has full column rank, then $\bar{\mathbf{C}} = \tilde{\mathbf{C}}\mathbf{U}$ for some nonsingular matrix \mathbf{U} .*

Proof. See de Almeida, Favier, and Mota [3], or Stegeman and de Almeida [41, Proposition 3.3]. \blacksquare

2.2. PARALIND uniqueness via CP uniqueness. Since a PARALIND decomposition $(\tilde{\mathbf{A}}\Psi, \tilde{\mathbf{B}}\Phi, \tilde{\mathbf{C}}\Omega)$ is a constrained CP decomposition, one might wonder whether CP uniqueness conditions can be used to prove the uniqueness of a PARALIND component matrix. The following example shows that CP uniqueness of $\tilde{\mathbf{A}}\Psi$ does not always imply uniqueness of $\tilde{\mathbf{A}}$ in PARALIND. Let $\Phi = \Omega = \mathbf{I}_3$ and

$$(2.10) \quad \Psi = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Let $\tilde{\mathbf{A}}, \tilde{\mathbf{B}}, \tilde{\mathbf{C}}$ have full column rank. The decomposition $(\tilde{\mathbf{A}}\Psi, \tilde{\mathbf{B}}, \tilde{\mathbf{C}})$, interpreted as a CP decomposition, is unique up to permutation and scaling since it satisfies Kruskal's uniqueness condition (1.3). The question is whether this implies that matrix $\tilde{\mathbf{A}}$ is unique up to permutation and scaling in the PARALIND decomposition $(\tilde{\mathbf{A}}\Psi, \tilde{\mathbf{B}}, \tilde{\mathbf{C}})$. Let an alternative PARALIND decomposition be given by $(\bar{\mathbf{A}}\Psi, \bar{\mathbf{B}}, \bar{\mathbf{C}})$. From the CP uniqueness, it follows that $\bar{\mathbf{A}}\Psi = \tilde{\mathbf{A}}\Psi\Pi\Lambda$, with Π a 3×3 permutation matrix, and Λ a nonsingular 3×3 diagonal matrix. If PARALIND uniqueness of $\tilde{\mathbf{A}}$ holds, then this should imply $\bar{\mathbf{A}} = \tilde{\mathbf{A}}\Pi_2\Lambda_2$, with Π_2 a 2×2 permutation matrix, and Λ_2 a nonsingular 2×2 diagonal matrix. However, the following example shows that this is not true. Let

$$(2.11) \quad \tilde{\mathbf{A}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \Pi = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \Lambda = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad \bar{\mathbf{A}} = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}.$$

Then it follows that

$$(2.12) \quad \bar{\mathbf{A}}\Psi = \tilde{\mathbf{A}}\Psi\Pi\Lambda = \begin{bmatrix} 0 & -1 & -1 \\ 1 & -1 & 0 \end{bmatrix}.$$

But the columns of $\tilde{\mathbf{A}}$ and $\bar{\mathbf{A}}$ are not equal up to permutation and scaling. Hence, matrix $\tilde{\mathbf{A}}$ is not unique in the PARALIND decomposition $(\tilde{\mathbf{A}}\Psi, \tilde{\mathbf{B}}, \tilde{\mathbf{C}})$ while the decomposition is unique when interpreted as a CP decomposition.

In general, for a decomposition $(\tilde{\mathbf{A}}\Psi, \tilde{\mathbf{B}}\Phi, \tilde{\mathbf{C}}\Omega)$, PARALIND uniqueness of $\tilde{\mathbf{A}}$ does follow from CP uniqueness of $\tilde{\mathbf{A}}\Psi$ if Ψ contains only columns of the $R_1 \times R_1$ identity matrix. In that case, the equality $\bar{\mathbf{A}}\Psi = \tilde{\mathbf{A}}\Psi\Pi\Lambda$ implies that the columns of $\bar{\mathbf{A}}$ are rescaled versions of the columns of $\tilde{\mathbf{A}}$. Since $\tilde{\mathbf{A}}$ has full column rank, it follows from

$$(2.13) \quad \tilde{\mathbf{A}} = \bar{\mathbf{A}} \Psi \Lambda^{-1} \Pi^T \Psi^T (\Psi \Psi^T)^{-1}$$

that also $\tilde{\mathbf{A}}$ has full column rank. Therefore, the relations between the columns of $\tilde{\mathbf{A}}$ and $\bar{\mathbf{A}}$ are one-to-one, and we have $\tilde{\mathbf{A}} = \bar{\mathbf{A}} \Pi_2 \Lambda_2$ for some permutation matrix Π_2 and a nonsingular diagonal matrix Λ_2 .

We use this fact to obtain PARALIND uniqueness conditions for $\tilde{\mathbf{A}}$ from the Kruskal-type CP uniqueness conditions by Guo et al. [15] for $\tilde{\mathbf{A}} \Psi$ with $k_{\tilde{\mathbf{A}} \Psi} = 1$.

THEOREM 2.5. *Let the PARALIND decomposition $(\tilde{\mathbf{A}} \Psi, \tilde{\mathbf{B}} \Phi, \tilde{\mathbf{C}} \Omega)$ be such that $\tilde{\mathbf{A}}, \tilde{\mathbf{B}}, \tilde{\mathbf{C}}$ have full column rank, and Ψ, Φ, Ω have full row rank and no all-zero columns. Also, let Ψ contain only columns of the $R_1 \times R_1$ identity matrix, and let $R_1 < R$. The component matrix $\tilde{\mathbf{A}}$ is unique up to permutation and scaling if one of the following holds:*

- (i) $\text{rank}(\Psi) + k_\Phi + k_\Omega \geq 2R + 2$,
- (ii) $k_\Phi < \text{rank}(\Phi)$, $k_\Omega < \text{rank}(\Omega)$, $\text{rank}(\Psi) + k_\Phi + k_\Omega \geq 2R + 1$,
- (iii) $\text{rank}(\Psi) + k_\Phi + k_\Omega + \max(\text{rank}(\Phi) - k_\Phi, \text{rank}(\Omega) - k_\Omega) \geq 2R + 2$,
 $\text{rank}(\Psi) + \min(k_\Phi, k_\Omega) \geq R + 2$.

Proof. Guo et al. [15, Theorems 2.1, 2.2, 2.3] show that conditions (i)–(iii) are sufficient for CP uniqueness up to permutation and scaling of $\tilde{\mathbf{A}} \Psi$ in the CP decomposition $(\tilde{\mathbf{A}} \Psi, \tilde{\mathbf{B}} \Phi, \tilde{\mathbf{C}} \Omega)$. Note that matrices $\tilde{\mathbf{A}}, \tilde{\mathbf{B}}, \tilde{\mathbf{C}}$ drop from conditions (i)–(iii) since they have full column rank. As argued above, the conditions on Ψ allow the CP uniqueness of $\tilde{\mathbf{A}} \Psi$ to be translated to PARALIND uniqueness of $\tilde{\mathbf{A}}$. \square

It was established in [15] that if $\tilde{\mathbf{A}}, \tilde{\mathbf{B}}, \tilde{\mathbf{C}}$ have full column rank, then the conditions of Theorem 2.5 are weaker than the ones of Theorem 2.2 when k_Φ and k_Ω are sufficiently large. The conditions of Theorem 2.5 are stronger than the ones of Theorem 2.2 for lower values of k_Φ and k_Ω .

3. How to check condition (2.9). Here, we discuss how condition (2.9) can be checked using (symbolic) linear algebra software. For each distinct value of $N_j^{(1)}$, we should determine the linearly independent vectors \mathbf{d}_j satisfying (2.9). We start with the case $N_j^{(1)} = 1$. A matrix has rank 1 if all its 2×2 submatrices have determinant zero, i.e., if all its 2×2 minors are zero. Let $\mathbf{d}_j = (\alpha_1 \dots \alpha_{R_1})^T$. Each entry of $\Phi \text{diag}(\Psi^T \mathbf{d}_j) \Omega^T$ is a linear function of $\alpha_1, \dots, \alpha_{R_1}$ with no constant term. As a result, a 2×2 minor of $\Phi \text{diag}(\Psi^T \mathbf{d}_j) \Omega^T$ is a second degree homogenous polynomial in $\alpha_1, \dots, \alpha_{R_1}$, or it is zero. The matrix $\Phi \text{diag}(\Psi^T \mathbf{d}_j) \Omega^T$ has size $R_2 \times R_3$. Hence, there are $R_2(R_2-1)R_3(R_3-1)/4$ determinants to check. Using symbolic computation software, we can write this as the linear system

$$(3.1) \quad \mathbf{U}^{(1)} \begin{pmatrix} \alpha_1 \alpha_2 \\ \vdots \\ \alpha_{R_1-1} \alpha_{R_1} \\ \alpha_1^2 \\ \vdots \\ \alpha_{R_1}^2 \end{pmatrix} = \mathbf{0},$$

where the matrix $\mathbf{U}^{(1)}$ has $R_2(R_2-1)R_3(R_3-1)/4$ rows and $R_1(R_1+1)/2$ columns. Each row in (3.1) corresponds to a distinct 2×2 minor. To obtain solutions for $\alpha_1, \dots, \alpha_{R_1}$, we can analyze the right null space of $\mathbf{U}^{(1)}$. For example, the MATLAB command `null(U1, 'r')` yields basis vectors for the right null space of $\mathbf{U}^{(1)}$ containing a lot of zero entries. If all basis vectors have zeros in the first $R_1(R_1-1)/2$ entries,

then $\alpha_s \alpha_t = 0$ for $s \neq t$. This implies $w(\mathbf{d}_j) \leq 1$. Analogously, the last R_1 entries of the basis vectors may be checked to obtain constraints of the form $\alpha_s^2 = 0$. Note that

$$(3.2) \quad \Phi \operatorname{diag}(\Psi^T \mathbf{d}_j) \Omega^T = \sum_{s=1}^{R_1} \alpha_s (\Phi \operatorname{diag}(\psi_s^T) \Omega^T).$$

Hence, if $N_s^{(1)} = \operatorname{rank}(\Phi \operatorname{diag}(\psi_s^T) \Omega^T) = 1$, then the term α_s^2 does not occur in the 2×2 minors of the matrix $\Phi \operatorname{diag}(\Psi^T \mathbf{d}_j) \Omega^T$.

For $N_j^{(1)} = 2$, condition (2.9) can be checked in the same way as above. Now all 3×3 minors of $\Phi \operatorname{diag}(\Psi^T \mathbf{d}_j) \Omega^T$ should be zero. Each minor is a third degree homogenous polynomial in $\alpha_1, \dots, \alpha_{R_1}$, or it is zero. Analogous to (3.1), we may build the linear system

$$(3.3) \quad \mathbf{U}^{(2)} \begin{pmatrix} \alpha_1 \alpha_2 \alpha_3 \\ \vdots \\ \alpha_{R_1-2} \alpha_{R_1-1} \alpha_{R_1} \\ \alpha_1^2 \alpha_2 \\ \vdots \\ \alpha_{R_1}^2 \alpha_{R_1-1} \\ \alpha_1^3 \\ \vdots \\ \alpha_{R_1}^3 \end{pmatrix} = \mathbf{0}.$$

The right null space of $\mathbf{U}^{(2)}$ may be analyzed to obtain solutions for $\alpha_1, \dots, \alpha_{R_1}$. It follows from (3.2) that the term α_s^3 does not occur in the 3×3 minors if $N_s^{(1)} \leq 2$. Also, the terms $\alpha_s^2 \alpha_t$, $s \neq t$, do not occur if $N_s^{(1)} = 1$.

For $N_j^{(1)} \geq 3$, an analogous method can be used. Whether or not it is convenient to analyze the minors of $\Phi \operatorname{diag}(\Psi^T \mathbf{d}_j) \Omega^T$ by means of the linear systems as above, or by writing out the minors, will depend on the complexity of the constraint matrices Ψ, Φ, Ω . When the values of $N_j^{(1)}$ are low and the constraint matrices contain many zero entries, building the complete linear system as above may not be convenient. This will become clear in the examples in the next section. Also, we will see that the constraints imposed on \mathbf{d}_j by condition (2.9) for low $N_j^{(1)}$ may imply additional constraints on \mathbf{d}_j for higher $N_j^{(1)}$. This is because the vectors $\mathbf{d}_1, \dots, \mathbf{d}_{R_1}$ should be linearly independent.

4. Examples of uniqueness in PARALIND. Here, we demonstrate the improvement when using Theorem 2.3 instead of Theorem 2.2. Also, the examples show that Theorem 2.3 may imply uniqueness in cases where Theorem 2.5 does not. In Example 1, matrix $\tilde{\mathbf{A}}$ is unique by Theorem 2.5 and Theorem 2.3 but not by Theorem 2.2. In Example 2, we prove the uniqueness of $\tilde{\mathbf{A}}, \tilde{\mathbf{B}}, \tilde{\mathbf{C}}$ by means of Theorem 2.3 and some further analysis. Theorems 2.2 and 2.5 cannot be used for this. In Example 3, we show that the particular form of alternatives $\tilde{\mathbf{C}}$ for $\tilde{\mathbf{C}} = \mathbf{I}_4$ can be understood by using Theorem 2.3. This is not the case for Theorem 2.2. To sum up, our examples demonstrate that Theorem 2.3 yields more insight into PARALIND uniqueness and the forms of alternative decompositions than Theorem 2.2. In our examples, we assume that $\tilde{\mathbf{A}}, \tilde{\mathbf{B}}, \tilde{\mathbf{C}}$ have full column rank.

Example 1. This example is taken from Guo et al. [15, Example 1]. We have $R_1 = R_2 = R_3 = 5$, $R = 6$, and

$$(4.1) \quad \Psi = [\mathbf{I}_5 \mathbf{e}], \quad \Phi = [\mathbf{I}_5 \mathbf{f}], \quad \Omega = [\mathbf{I}_5 \mathbf{g}],$$

where $\mathbf{e} = (0 \ 0 \ 0 \ 0 \ 1)^T$, $\mathbf{f} = (1 \ 1 \ 1 \ 1 \ 1)^T$, and $\mathbf{g} = (1 \ 1 \ 1 \ 1 \ 0)^T$. Since $\text{rank}(\Psi) = 5$, $k_\Phi = 5$, and $k_\Omega = 4$, matrix $\tilde{\mathbf{A}}$ is unique up to permutation and scaling by Theorem 2.5 (i). In [15] it was verified that condition (2.8) of Theorem 2.2 does not hold. Next, we show that condition (2.9) of the new Theorem 2.3 does hold.

We leave it to the reader to verify that $\mathbf{G}_1, \mathbf{G}_2, \mathbf{G}_3$ have full column rank, and that $N_1^{(1)} = N_2^{(1)} = N_3^{(1)} = N_4^{(1)} = 1$ and $N_5^{(1)} = 2$. First, we check condition (2.9) for $N_j^{(1)} = 1$. We have

$$(4.2) \quad \Phi \text{diag}(\Psi^T \mathbf{d}_j) \Omega^T = \begin{bmatrix} \alpha_1 + \alpha_5 & \alpha_5 & \alpha_5 & \alpha_5 & 0 \\ \alpha_5 & \alpha_2 + \alpha_5 & \alpha_5 & \alpha_5 & 0 \\ \alpha_5 & \alpha_5 & \alpha_3 + \alpha_5 & \alpha_5 & 0 \\ \alpha_5 & \alpha_5 & \alpha_5 & \alpha_4 + \alpha_5 & 0 \\ \alpha_5 & \alpha_5 & \alpha_5 & \alpha_5 & \alpha_5 \end{bmatrix},$$

with $\mathbf{d}_j = (\alpha_1 \ \alpha_2 \ \alpha_3 \ \alpha_4 \ \alpha_5)^T$. By using symbolic computations in MATLAB, we construct the matrix $\mathbf{U}^{(1)}$ in (3.1) from the distinct 2×2 minors of $\Phi \text{diag}(\Psi^T \mathbf{d}_j) \Omega^T$. Analyzing the right null space of $\mathbf{U}^{(1)}$ with the MATLAB command `null(U1, 'r')` yields the following constraints: $\alpha_s \alpha_t = 0$, $s \neq t$, and $\alpha_5^2 = 0$. Since $N_1^{(1)} = N_2^{(1)} = N_3^{(1)} = N_4^{(1)} = 1$, we need to pick four linearly independent vectors $\mathbf{d}_1, \dots, \mathbf{d}_4$ satisfying these constraints. Each vector can have at most one nonzero entry, and the last (fifth) entry is zero in all four vectors, which implies that $\mathbf{d}_1, \dots, \mathbf{d}_4$ are equal to the first four columns of \mathbf{I}_5 up to permutation and scaling.

The case $N_j^{(1)} = 2$ corresponds to a fifth vector \mathbf{d}_5 . Since $\mathbf{d}_1, \dots, \mathbf{d}_4$ have their fifth (last) entry equal to zero, it follows that $\alpha_5 \neq 0$ in \mathbf{d}_5 in order to have five linearly independent vectors $\mathbf{d}_1, \dots, \mathbf{d}_5$. We denote a 3×3 minor of $\Phi \text{diag}(\Psi^T \mathbf{d}_j) \Omega^T$ as $M_{(uvw,xyz)}$, where rows u, v, w and columns x, y, z are included. We have

$$(4.3) \quad M_{(125,145)} = \alpha_1 \alpha_5^2, \quad M_{(235,245)} = \alpha_2 \alpha_5^2, \quad M_{(235,345)} = -\alpha_3 \alpha_5^2, \quad M_{(345,245)} = \alpha_4 \alpha_5^2.$$

Since $\alpha_5 \neq 0$, it follows that $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$. Hence, vector \mathbf{d}_5 is equal to the fifth column of \mathbf{I}_5 up to scaling. This implies that $[\mathbf{d}_1 \ \dots \ \mathbf{d}_5] = \mathbf{\Pi} \mathbf{\Lambda}$, and condition (2.9) holds.

Example 2. The next example is taken from [41, section 8]. Let $R_1 = R_2 = R_3 = 3$, let $R = 4$, and let Ψ, Φ, Ω be given by (1.5). In [41] it is claimed that uniqueness of $\tilde{\mathbf{A}}, \tilde{\mathbf{B}}, \tilde{\mathbf{C}}$ follows from rewriting the PARALIND decomposition as Tucker3 with core (1.7) and applying a result from Kiers, Ten Berge, and Rocci [20] on Tucker3 uniqueness. This claim is false, however. The result from [20] does not allow certain Tucker3 core entries to be set to zero, which is done in this choice of (Ψ, Φ, Ω) (equation (1.8) gives the core array for this example). However, by using Theorem 2.3 and some further analysis, we show that matrices $\tilde{\mathbf{A}}, \tilde{\mathbf{B}}, \tilde{\mathbf{C}}$ are indeed unique up to permutation and scaling. Theorem 2.2 cannot be used for this.

First, we consider uniqueness of $\tilde{\mathbf{A}}$. We have $\text{rank}(\Psi) = 3$, $k_\Phi = 1$, and $k_\Omega = 1$. Conditions (i)–(iii) of Theorem 2.5 do not hold. Next, we consider condition (2.8) of Theorem 2.2. It can be verified that $\mathbf{G}_1, \mathbf{G}_2, \mathbf{G}_3$ have full column rank, and that $N_1^{(1)} = 2$ and $N_2^{(1)} = N_3^{(1)} = 1$. We have

$$(4.4) \quad \Phi \operatorname{diag}(\Psi^T \mathbf{d}_j) \Omega^T = \begin{bmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & \alpha_1 \\ 0 & 0 & \alpha_3 \end{bmatrix},$$

with $\mathbf{d}_j = (\alpha_1 \ \alpha_2 \ \alpha_3)^T$. Since this matrix has rank 2 if $\alpha_1 \neq 0$, $\alpha_2 \neq 0$, and $\alpha_3 = 0$, condition (2.8) does not hold. Finally, we turn to condition (2.9) of Theorem 2.3. For $N_j^{(1)} = 1$, analysis of the right null space of $\mathbf{U}^{(1)}$ (or writing out the 2×2 minors by hand) yields the constraints $\alpha_s \alpha_t = 0$, $s \neq t$, and $\alpha_1^2 = 0$. Since we need two linearly independent vectors $\mathbf{d}_2, \mathbf{d}_3$ satisfying these constraints, it follows that they are equal to columns 2 and 3 of \mathbf{I}_3 up to permutation and scaling. For $N_j^{(1)} = 2$, we need a third vector \mathbf{d}_1 such that $\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3$ are linearly independent. It follows that $\alpha_1 \neq 0$ for \mathbf{d}_1 . There is only one 3×3 minor of $\Phi \operatorname{diag}(\Psi^T \mathbf{d}_j) \Omega^T$ (the determinant of the matrix itself), which reads as $\alpha_1 \alpha_2 \alpha_3$. Since $\alpha_1 \neq 0$, we obtain $\alpha_2 \alpha_3 = 0$. We have uniqueness of $\tilde{\mathbf{A}}$ if $\alpha_2 = \alpha_3 = 0$.

To prove this, we return to equating the PARALIND decomposition to its alternative. Since $\tilde{\mathbf{B}}$ and $\tilde{\mathbf{C}}$ have full column rank, we set $\tilde{\mathbf{B}} = \tilde{\mathbf{C}} = \mathbf{I}_3$ without loss of generality; see [41, lemma 3.4]. As in (A.4), we write

$$(4.5) \quad \Phi \operatorname{diag}(\Psi^T \mathbf{d}_j) \Omega^T = \tilde{\mathbf{B}} \Phi \operatorname{diag}(\psi_j^T) \Omega^T \tilde{\mathbf{C}}^T, \quad j = 1, 2, 3.$$

Let $\mathbf{d}_2 = (0 \ * \ 0)^T$ and $\mathbf{d}_3 = (0 \ 0 \ *)^T$, where $*$ denotes a nonzero entry. For $j = 2$, we use (4.4) and (4.5) to obtain

$$(4.6) \quad \Phi \operatorname{diag}(\Psi^T \mathbf{d}_2) \Omega^T = \begin{bmatrix} 0 & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & 0 \end{bmatrix} = \bar{\mathbf{b}}_2 \bar{\mathbf{c}}_2^T,$$

which implies that $\bar{\mathbf{b}}_2 = (0 \ * \ 0)^T$ and $\bar{\mathbf{c}}_2 = (0 \ * \ 0)^T$. Analogously, for $j = 3$ we get

$$(4.7) \quad \Phi \operatorname{diag}(\Psi^T \mathbf{d}_3) \Omega^T = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & * \end{bmatrix} = \bar{\mathbf{b}}_3 \bar{\mathbf{c}}_3^T,$$

which implies $\bar{\mathbf{b}}_3 = (0 \ 0 \ *)^T$ and $\bar{\mathbf{c}}_3 = (0 \ 0 \ *)^T$. Finally, for $j = 1$ we have

$$(4.8) \quad \Phi \operatorname{diag}(\Psi^T \mathbf{d}_1) \Omega^T = \begin{bmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & \alpha_1 \\ 0 & 0 & \alpha_3 \end{bmatrix} = \bar{\mathbf{b}}_1 \bar{\mathbf{c}}_1^T + \bar{\mathbf{b}}_2 \bar{\mathbf{c}}_2^T,$$

with $\alpha_1 \neq 0$ and $\alpha_2 \alpha_3 = 0$. It follows from the above that matrix $\bar{\mathbf{b}}_2 \bar{\mathbf{c}}_3^T$ has only its (2,3)-entry nonzero. Hence, the rank-1 matrix $\bar{\mathbf{b}}_1 \bar{\mathbf{c}}_1^T$ is equal to the left-hand side of (4.8) with the (2,3)-entry having any convenient value. For $\alpha_1 \neq 0$, it follows that $\alpha_2 = \alpha_3 = 0$ must hold.

Uniqueness of $\tilde{\mathbf{B}}$ and $\tilde{\mathbf{C}}$ follows analogously by interchanging the roles of $\tilde{\mathbf{A}}\Psi$, $\tilde{\mathbf{B}}\Phi$, and $\tilde{\mathbf{C}}\Omega$.

Example 3. This example is taken from Bro et al. [4, section 3.2.5]. We have $R_1 = 3$, $R_2 = 6$, $R_3 = 4$, $R = 6$, $\Phi = \mathbf{I}_6$, and

$$(4.9) \quad \Psi = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}, \quad \Omega = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}.$$

In Stegeman and de Almeida [41, section 5] it was verified that $\tilde{\mathbf{A}}$ is unique by condition (2.8). Since $\text{rank}(\Psi) = 3$, $k_\Phi = \text{rank}(\Phi) = 6$, and $k_\Omega = \text{rank}(\Omega) = 4$, conditions (i)–(iii) of Theorem 2.5 do not hold.

Numerical experiments yield alternatives for $\tilde{\mathbf{C}} = \mathbf{I}_4$ of the form

$$(4.10) \quad \bar{\mathbf{C}} = \begin{bmatrix} * & 0 & 0 & 0 \\ 0 & * & 0 & 0 \\ 0 & 0 & * & 0 \\ * & * & * & * \end{bmatrix},$$

where $*$ denotes a nonzero element. Hence, only the last column of $\tilde{\mathbf{C}}$ appears to be identified up to scaling. Or one could say that only the first three rows of $\tilde{\mathbf{C}}$ are identified. In [41, section 10], it was stated that this type of uniqueness is not understood from condition (2.8). By making use of the new condition (2.9), however, we are able to explain this form of $\bar{\mathbf{C}}$. First, we translate condition (2.9) to $\tilde{\mathbf{C}}$. It can be verified that $\mathbf{G}_1, \mathbf{G}_2, \mathbf{G}_3$ have full column rank, and that $N_1^{(3)} = N_2^{(3)} = N_3^{(3)} = 1$ and $N_4^{(3)} = 3$. We have

$$(4.11) \quad \Psi \text{diag}(\Omega^T \mathbf{d}_j) \Phi^T = \begin{bmatrix} \alpha_1 & \alpha_4 - \alpha_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha_2 & \alpha_4 - \alpha_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha_3 & \alpha_4 - \alpha_3 \end{bmatrix},$$

with $\mathbf{d}_j = (\alpha_1 \ \alpha_2 \ \alpha_3 \ \alpha_4)^T$. Analysis of the 2×2 minors shows that this matrix has rank 1 only if $\alpha_s \alpha_t = 0$ for $s \neq t$, and only if $\alpha_4 = 0$. Since we need three linearly independent vectors $\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3$ satisfying these constraints, it follows that they are equal to the first three columns of \mathbf{I}_4 up to permutation and scaling. The matrix has to have rank $N_4^{(3)} = 3$ for a vector \mathbf{d}_4 that is linearly independent from $\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3$. This implies $\alpha_4 \neq 0$ in \mathbf{d}_4 . The entries $\alpha_1, \alpha_2, \alpha_3$ can be nonzero as well. Hence, we have constrained the alternative to

$$(4.12) \quad \bar{\mathbf{C}}^{-T} = \begin{bmatrix} * & 0 & 0 & * \\ 0 & * & 0 & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{bmatrix},$$

where the order of columns may be different (this also depends on the uniqueness properties of \mathbf{A} and $\tilde{\mathbf{B}}$). For a nonsingular matrix \mathbf{X} , we use the notation $\mathbf{X}^{-T} = (\mathbf{X}^{-1})^T = (\mathbf{X}^T)^{-1}$. For the form of $\bar{\mathbf{C}}^{-T}$ in (4.12), we obtain a $\bar{\mathbf{C}}$ exactly of the form (4.10). Hence, three unique columns of $\bar{\mathbf{C}}^{-T}$ translate into three unique rows of $\bar{\mathbf{C}}$.

5. Partial uniqueness conditions for PARALIND. Apart from uniqueness up to permutation and scaling, other types of uniqueness are also encountered in PARALIND decompositions. In this section, we present so-called partial uniqueness conditions, where we call component matrix $\tilde{\mathbf{B}}$ partially unique if its columns can be partitioned into disjoint subsets and each subset is unique up to a nonsingular transformation. This definition is used in Bro et al. [4] and Stegeman and de Almeida [41]. For partial uniqueness in CP, see Ten Berge [42]. Formally, we define partial uniqueness in PARALIND as follows.

DEFINITION 5.1. *Let the PARALIND decomposition $(\tilde{\mathbf{A}}\Psi, \tilde{\mathbf{B}}\Phi, \tilde{\mathbf{C}}\Omega)$ be such that $\tilde{\mathbf{A}}, \tilde{\mathbf{B}}, \tilde{\mathbf{C}}$ have full column rank, and let Ψ, Φ, Ω have full row rank and no all-zero columns. Let the columns of $\tilde{\mathbf{B}}$ be partitioned into disjoint subsets as $\tilde{\mathbf{B}}\Pi_b = [\tilde{\mathbf{B}}_1 | \dots | \tilde{\mathbf{B}}_F]$, where Π_b is a permutation matrix. We call $\tilde{\mathbf{B}}$ partially unique with*

respect to Π_b if, for any alternative PARALIND decomposition $(\bar{\mathbf{A}}\Psi, \bar{\mathbf{B}}\Phi, \bar{\mathbf{C}}\Omega)$ with $\bar{\mathbf{B}}$ partitioned as $\bar{\mathbf{B}}\Pi_b = [\bar{\mathbf{B}}_1 | \dots | \bar{\mathbf{B}}_F]$, it holds that $\bar{\mathbf{B}}_f = \tilde{\mathbf{B}}_{\pi(f)} \mathbf{S}_f$, $f = 1, \dots, F$, with $\pi(\cdot)$ a unique permutation on $\{1, \dots, F\}$, and \mathbf{S}_f a unique nonsingular matrix.

Note that when checking partial uniqueness of $\tilde{\mathbf{B}}$ one is free to choose a suitable column permutation Π_b and partition of $\tilde{\mathbf{B}}\Pi_b$.

In section 5.1, we present the partial uniqueness condition of [41] as well as our improved version of it. The relation between these two conditions is analogous to the relation between Theorems 2.2 and 2.3. In section 5.2, we discuss an alternative approach to obtain a partial uniqueness condition based on a result for CP by Guo et al. [15].

5.1. Partial uniqueness conditions for one PARALIND component matrix. The partial uniqueness condition of [41] is as follows. For a partitioned vector $\mathbf{x} = (\mathbf{x}_1^T | \dots | \mathbf{x}_F^T)^T$, let $w'(\mathbf{x})$ denote the number of parts of \mathbf{x} that are not all-zero. For a vector \mathbf{g}_f partitioned as \mathbf{x} , with $w'(\mathbf{g}_f) = 1$ and generic¹ entries in part f , let

$$(5.1) \quad M_f^{(2)} = \text{rank}(\Omega \text{diag}(\Phi^T \Pi_b \mathbf{g}_f) \Psi^T), \quad f = 1, \dots, F.$$

THEOREM 5.2. Let the PARALIND decomposition $(\tilde{\mathbf{A}}\Psi, \tilde{\mathbf{B}}\Phi, \tilde{\mathbf{C}}\Omega)$ be such that $\tilde{\mathbf{A}}, \tilde{\mathbf{B}}, \tilde{\mathbf{C}}$ have full column rank, and Ψ, Φ, Ω have full row rank and no all-zero columns. Let the columns of $\tilde{\mathbf{B}}$ be partitioned as $\tilde{\mathbf{B}}\Pi_b = [\tilde{\mathbf{B}}_1 | \dots | \tilde{\mathbf{B}}_F]$, where Π_b is a permutation matrix. If \mathbf{G}_2 has full column rank, and if for any nonzero vector \mathbf{d} with the same partition as $\tilde{\mathbf{B}}\Pi_b$,

$$(5.2) \quad \text{rank}(\Omega \text{diag}(\Phi^T \Pi_b \mathbf{d}) \Psi^T) \leq \max(M_1^{(2)}, \dots, M_F^{(2)}) \quad \text{implies} \quad w'(\mathbf{d}) = 1,$$

then $\tilde{\mathbf{B}}$ is partially unique with respect to Π_b as defined in Definition 5.1.

Proof. See Stegeman and de Almeida [41, section 6]. See Appendix B for a simpler proof. \square

When taking the column blocks of $\tilde{\mathbf{B}}$ equal to its columns, i.e., setting $\Pi_b = \mathbf{I}_{R_2}$ and $F = R_2$, Theorem 5.2 reduces to Theorem 2.2 translated to $\tilde{\mathbf{B}}$. We prove the following improved version of Theorem 5.2. For a partitioned vector $\mathbf{x} = (\mathbf{x}_1^T | \dots | \mathbf{x}_F^T)^T$, let $\mathbf{x}^{(c)}$ be an $F \times 1$ vector with $x_f^{(c)} = 0$ if $\mathbf{x}_f = \mathbf{0}$, and $x_f^{(c)} = 1$ if $\mathbf{x}_f \neq \mathbf{0}$, $f = 1, \dots, F$.

THEOREM 5.3. Let the PARALIND decomposition $(\tilde{\mathbf{A}}\Psi, \tilde{\mathbf{B}}\Phi, \tilde{\mathbf{C}}\Omega)$ be such that $\tilde{\mathbf{A}}, \tilde{\mathbf{B}}, \tilde{\mathbf{C}}$ have full column rank, and Ψ, Φ, Ω have full row rank and no all-zero columns. Let the columns of $\tilde{\mathbf{B}}$ be partitioned as $\tilde{\mathbf{B}}\Pi_b = [\tilde{\mathbf{B}}_1 | \dots | \tilde{\mathbf{B}}_F]$, where Π_b is a permutation matrix. If $\mathbf{G}_1, \mathbf{G}_2, \mathbf{G}_3$ have full column rank, and if for any set of vectors $\mathbf{d}_1, \dots, \mathbf{d}_F$ with the same partition as $\tilde{\mathbf{B}}\Pi_b$, and $\mathbf{d}_1^{(c)}, \dots, \mathbf{d}_F^{(c)}$ linearly independent,

$$(5.3) \quad \text{rank}(\Omega \text{diag}(\Phi^T \Pi_b \mathbf{d}_f) \Psi^T) = M_f^{(2)}, \quad f = 1, \dots, F, \quad \text{implies} \\ w'(\mathbf{d}_f) = 1, \quad f = 1, \dots, F,$$

then $\tilde{\mathbf{B}}$ is partially unique with respect to Π_b as defined in Definition 5.1.

Proof. See Appendix B for the proof. \square

¹We mean the following. The entries in part f of vector \mathbf{g}_f are randomly sampled from a continuous probability distribution, and the rank in (5.1) occurs with probability one. This rank value is equal to the maximal rank over all \mathbf{g}_f with nonzero entries only in part f .

The proofs of Theorems 5.2 and 5.3 are in Appendix B, and are similar. Our proof of Theorem 5.2 is simpler than in [41], since we do not use De Lathauwer's [10] equivalence lemma for partitioned matrices.

It can be seen that condition (5.2) implies condition (5.3). Hence, if not only \mathbf{G}_1 but also \mathbf{G}_2 and \mathbf{G}_3 have full column rank, then Theorem 5.3 is a relaxation of Theorem 5.2.

To obtain partial uniqueness conditions for $\tilde{\mathbf{A}}$ or $\tilde{\mathbf{C}}$, we can define ranks $M_f^{(1)}$ and $M_f^{(3)}$ analogous to (5.1), and interchange the roles of $\tilde{\mathbf{A}}\Psi$, $\tilde{\mathbf{B}}\Phi$, $\tilde{\mathbf{C}}\Omega$, and $\mathbf{G}_1, \mathbf{G}_2, \mathbf{G}_3$, and $M_f^{(1)}, M_f^{(2)}, M_f^{(3)}$ in Theorems 5.2 and 5.3.

5.2. PARALIND partial uniqueness via partitioning in CP. Here, we discuss a different approach to obtain partial uniqueness results. For a CP decomposition $(\mathbf{A}, \mathbf{B}, \mathbf{C})$, Guo et al. [15] proved that if one component matrix, say \mathbf{A} , is unique up to permutation and scaling, then the decomposition can be split up into a sum of smaller subdecompositions, and the uniqueness properties of \mathbf{B} (and \mathbf{C}) can be studied for each subdecomposition separately. Below, we show that in a PARALIND decomposition $(\tilde{\mathbf{A}}\Psi, \tilde{\mathbf{B}}\Phi, \tilde{\mathbf{C}}\Omega)$ this result yields a partial uniqueness condition for $\tilde{\mathbf{B}}$ (or $\tilde{\mathbf{C}}$) if $\tilde{\mathbf{A}}$ is unique up to permutation and scaling. Let

$$(5.4) \quad \tilde{\mathbf{A}}\Psi\mathbf{\Pi}_a = [\tilde{\mathbf{A}}_1 | \dots | \tilde{\mathbf{A}}_F],$$

where $\mathbf{\Pi}_a$ is a permutation matrix. The partition must be such that

$$(5.5) \quad \text{span}(\tilde{\mathbf{A}}) = \text{span}(\tilde{\mathbf{A}}_1) \oplus \dots \oplus \text{span}(\tilde{\mathbf{A}}_F),$$

where \oplus denotes the direct sum of the subspaces. Hence,

$$(5.6) \quad \text{span}(\tilde{\mathbf{A}}_f) \cap \left(\bigcup_{g \neq f} \text{span}(\tilde{\mathbf{A}}_g) \right) = \{\mathbf{0}\}, \quad f = 1, \dots, F.$$

Let $\tilde{\mathbf{B}}$ and $\tilde{\mathbf{C}}$ be partitioned as

$$(5.7) \quad \tilde{\mathbf{B}}\Phi\mathbf{\Pi}_a = [\tilde{\mathbf{B}}_1 | \dots | \tilde{\mathbf{B}}_F] \quad \text{and} \quad \tilde{\mathbf{C}}\Omega\mathbf{\Pi}_a = [\tilde{\mathbf{C}}_1 | \dots | \tilde{\mathbf{C}}_F],$$

where $\tilde{\mathbf{A}}_f$, $\tilde{\mathbf{B}}_f$, and $\tilde{\mathbf{C}}_f$ have the same number of columns, $f = 1, \dots, F$. Note that the same permutation matrix $\mathbf{\Pi}_a$ is used in (5.4) and (5.7). The subdecompositions $(\tilde{\mathbf{A}}_f, \tilde{\mathbf{B}}_f, \tilde{\mathbf{C}}_f)$, $f = 1, \dots, F$, are also of PARALIND form. Indeed, we have $\tilde{\mathbf{A}}_f = \tilde{\mathbf{A}}_f\Psi_f$, $\tilde{\mathbf{B}}_f = \tilde{\mathbf{B}}_f\Phi_f$, and $\tilde{\mathbf{C}}_f = \tilde{\mathbf{C}}_f\Omega_f$, where Ψ_f , Φ_f , and Ω_f are smaller constraint matrices. We define $\hat{\mathbf{A}}_f, \hat{\mathbf{B}}_f, \hat{\mathbf{C}}_f$ as the subsets of columns of $\tilde{\mathbf{A}}, \tilde{\mathbf{B}}, \tilde{\mathbf{C}}$, respectively, that occur in the linear combinations that constitute the columns of $\tilde{\mathbf{A}}_f, \tilde{\mathbf{B}}_f, \tilde{\mathbf{C}}_f$. Then the constraint matrices Ψ_f, Φ_f, Ω_f are uniquely defined due to the full column rank of $\hat{\mathbf{A}}_f, \hat{\mathbf{B}}_f, \hat{\mathbf{C}}_f$.

Next, we formulate our PARALIND partial uniqueness condition for $\tilde{\mathbf{B}}$ under the condition that $\Psi\mathbf{\Pi}_a = \mathbf{I}_{R_1} \otimes \mathbf{1}_n^T$, with $\mathbf{1}_n$ a vector of n ones. In that case, $\tilde{\mathbf{A}}_f = [\tilde{\mathbf{a}}_f \dots \tilde{\mathbf{a}}_f]$ with column $\tilde{\mathbf{a}}_f$ repeated n times, $f = 1, \dots, R_1$. Note that $F = R_1$ here.

THEOREM 5.4. *Let the PARALIND decomposition $(\tilde{\mathbf{A}}\Psi, \tilde{\mathbf{B}}\Phi, \tilde{\mathbf{C}}\Omega)$ be such that $\tilde{\mathbf{A}}, \tilde{\mathbf{B}}, \tilde{\mathbf{C}}$ have full column rank, and Ψ, Φ, Ω have full row rank and no all-zero columns. Let \mathbf{A} be unique up to permutation and scaling, and let the component matrices be partitioned as in (5.4)–(5.7) for some permutation matrix $\mathbf{\Pi}_a$. Also, let $\Psi\mathbf{\Pi}_a = \mathbf{I}_F \otimes \mathbf{1}_n^T$. If there exists a permutation $\mathbf{\Pi}_b$ such that $\tilde{\mathbf{B}}\mathbf{\Pi}_b = [\hat{\mathbf{B}}_1 | \dots | \hat{\mathbf{B}}_F]$, and $(\Omega_f \odot \Psi_f)\Phi_f^T$*

has full column rank for $f = 1, \dots, F$, then $\tilde{\mathbf{B}}$ is partially unique with respect to $\mathbf{\Pi}_b$ as defined in Definition 5.1.

Proof. We consider the PARALIND decomposition $(\tilde{\mathbf{A}}\mathbf{\Psi}\mathbf{\Pi}_a, \tilde{\mathbf{B}}\mathbf{\Phi}\mathbf{\Pi}_a, \tilde{\mathbf{C}}\mathbf{\Omega}\mathbf{\Pi}_a)$. According to Definition 5.1, to prove partial uniqueness of $\tilde{\mathbf{B}}$ with respect to $\mathbf{\Pi}_b$ we should consider the F column blocks $\tilde{\mathbf{B}}_1, \dots, \tilde{\mathbf{B}}_F$ of $\tilde{\mathbf{B}}\mathbf{\Pi}_b$. Due to the requirement on $\mathbf{\Pi}_b$ in the statement of the theorem, we have

$$\begin{aligned}
 \tilde{\mathbf{B}}\mathbf{\Phi}\mathbf{\Pi}_a &= [\tilde{\mathbf{B}}_1 | \dots | \tilde{\mathbf{B}}_F] \\
 &= [\hat{\mathbf{B}}_1\mathbf{\Phi}_1 | \dots | \hat{\mathbf{B}}_F\mathbf{\Phi}_F] \\
 (5.8) \quad &= [\hat{\mathbf{B}}_1 | \dots | \hat{\mathbf{B}}_F] \text{ blockdiag}(\mathbf{\Phi}_1, \dots, \mathbf{\Phi}_F) \\
 &= \tilde{\mathbf{B}}\mathbf{\Pi}_b \text{ blockdiag}(\mathbf{\Phi}_1, \dots, \mathbf{\Phi}_F).
 \end{aligned}$$

Since $\tilde{\mathbf{B}}$ has full column rank, it follows that $\mathbf{\Phi}\mathbf{\Pi}_a = \mathbf{\Pi}_b \text{ blockdiag}(\mathbf{\Phi}_1, \dots, \mathbf{\Phi}_F) = \mathbf{\Pi}_b\mathbf{\Phi}^*$, where the last equality defines $\mathbf{\Phi}^*$. Hence, we consider the PARALIND decomposition $(\tilde{\mathbf{A}}\mathbf{\Psi}\mathbf{\Pi}_a, \tilde{\mathbf{B}}\mathbf{\Pi}_b\mathbf{\Phi}^*, \tilde{\mathbf{C}}\mathbf{\Omega}\mathbf{\Pi}_a)$.

We denote an alternative decomposition by $(\bar{\mathbf{A}}\mathbf{\Psi}\mathbf{\Pi}_a, \bar{\mathbf{B}}\mathbf{\Pi}_b\mathbf{\Phi}^*, \bar{\mathbf{C}}\mathbf{\Omega}\mathbf{\Pi}_a)$. Equating the mode-1 matrix unfoldings of the two decompositions yields

$$(5.9) \quad (\tilde{\mathbf{B}}\mathbf{\Pi}_b\mathbf{\Phi}^* \odot \tilde{\mathbf{C}}\mathbf{\Omega}\mathbf{\Pi}_a)(\tilde{\mathbf{A}}\mathbf{\Psi}\mathbf{\Pi}_a)^T = (\bar{\mathbf{B}}\mathbf{\Pi}_b\mathbf{\Phi}^* \odot \bar{\mathbf{C}}\mathbf{\Omega}\mathbf{\Pi}_a)(\bar{\mathbf{A}}\mathbf{\Psi}\mathbf{\Pi}_a)^T.$$

Since $\tilde{\mathbf{A}}$ is unique up to permutation and scaling, it follows that $\bar{\mathbf{A}} = \tilde{\mathbf{A}}\mathbf{\Pi}_2\mathbf{\Lambda}_2$ for some permutation matrix $\mathbf{\Pi}_2$ and nonsingular diagonal matrix $\mathbf{\Lambda}_2 = \text{diag}(\lambda_1, \dots, \lambda_{R_1})$. From the assumed form of $\mathbf{\Psi}\mathbf{\Pi}_a$, it follows that

$$(5.10) \quad \bar{\mathbf{A}}\mathbf{\Psi}\mathbf{\Pi}_a = \tilde{\mathbf{A}}((\mathbf{\Pi}_2\mathbf{\Lambda}_2) \otimes \mathbf{1}_n^T) = [\lambda_{\pi_2(1)}\tilde{\mathbf{A}}_{\pi_2(1)} | \dots | \lambda_{\pi_2(R_1)}\tilde{\mathbf{A}}_{\pi_2(R_1)}],$$

where π_2 denotes the permutation defined by $\mathbf{\Pi}_2$. Hence, the permutation and rescaling of the columns of \mathbf{A} translates into the permutation and rescaling of the parts $\tilde{\mathbf{A}}_f$ of $\tilde{\mathbf{A}}\mathbf{\Psi}\mathbf{\Pi}_a$. Let $\tilde{\mathbf{B}}\mathbf{\Pi}_b\mathbf{\Phi}^* = [\tilde{\mathbf{B}}_1 | \dots | \tilde{\mathbf{B}}_F]$ and $\tilde{\mathbf{C}}\mathbf{\Omega}\mathbf{\Pi}_a = [\tilde{\mathbf{C}}_1 | \dots | \tilde{\mathbf{C}}_F]$ be partitioned in the same way as (5.7). The smaller PARALIND decompositions of each part f for the original and alternative decompositions are given by $\tilde{\mathbf{A}}_f = \hat{\mathbf{A}}_f\mathbf{\Psi}_f$, $\tilde{\mathbf{B}}_f = \hat{\mathbf{B}}_f\mathbf{\Phi}_f$, $\tilde{\mathbf{C}}_f = \hat{\mathbf{C}}_f\mathbf{\Omega}_f$, and $\bar{\mathbf{B}}_f = \hat{\mathbf{B}}_f\mathbf{\Phi}_f$, $\bar{\mathbf{C}}_f = \hat{\mathbf{C}}_f\mathbf{\Omega}_f$. As in Guo et al. [15, Theorem 3.1], equations (5.9)–(5.10) and the requirement (5.5) imply that (5.9) is equivalent to equating the decompositions of the F parts separately:

$$(5.11) \quad (\tilde{\mathbf{B}}_f \odot \tilde{\mathbf{C}}_f)\tilde{\mathbf{A}}_f^T = (\bar{\mathbf{B}}_{\tilde{\pi}_2(f)} \odot \bar{\mathbf{C}}_{\tilde{\pi}_2(f)})(\lambda_{\tilde{\pi}_2(f)}\tilde{\mathbf{A}}_f)^T, \quad f = 1, \dots, F,$$

where $\tilde{\pi}_2$ denotes the inverse of the permutation π_2 . In terms of the smaller PARALIND decompositions, (5.11) is written as

$$\begin{aligned}
 (5.12) \quad &(\hat{\mathbf{B}}_f \otimes \hat{\mathbf{C}}_f)(\mathbf{\Phi}_f \odot \mathbf{\Omega}_f)\mathbf{\Psi}_f^T \hat{\mathbf{A}}_f^T = (\hat{\mathbf{B}}_{\tilde{\pi}_2(f)} \otimes \hat{\mathbf{C}}_{\tilde{\pi}_2(f)})(\mathbf{\Phi}_{\tilde{\pi}_2(f)} \odot \mathbf{\Omega}_{\tilde{\pi}_2(f)})\mathbf{\Psi}_f^T (\lambda_{\tilde{\pi}_2(f)}\hat{\mathbf{A}}_f)^T, \\
 &f = 1, \dots, F.
 \end{aligned}$$

Due to the requirement on $\mathbf{\Pi}_b$, we have $\tilde{\mathbf{B}}\mathbf{\Pi}_b = [\hat{\mathbf{B}}_1 | \dots | \hat{\mathbf{B}}_F]$. Next, we want to apply Proposition 2.4(ii) to each subdecomposition in (5.12) separately. Therefore, we need to have the same constraint matrices on both sides of (5.12). This implies that the permutation $\mathbf{\Pi}_2$ must be such that the alternative on the right-hand side of (5.12) has the same $\mathbf{\Phi}_f$ and $\mathbf{\Omega}_f$ as on the left-hand side of (5.12). Note that due to the assumed form of $\mathbf{\Psi}\mathbf{\Pi}_a$, we have $\hat{\mathbf{A}}_f = [\tilde{\mathbf{a}}_f]$ and $\mathbf{\Psi}_f = \mathbf{1}_n^T$.

For the subset of Π_2 satisfying this requirement, partial uniqueness of $\tilde{\mathbf{B}}\Pi_b = [\tilde{\mathbf{B}}_1|\dots|\tilde{\mathbf{B}}_F]$ follows from applying Proposition 2.4(ii) to each subdecomposition in (5.12) separately. Since $\tilde{\mathbf{A}}_f, \tilde{\mathbf{B}}_f, \tilde{\mathbf{C}}_f$ have full column rank, partial uniqueness follows if $(\Omega_f \odot \Psi_f)\Phi_f^T$ has full column rank for $f = 1, \dots, F$. \square

Theorem 5.4 yields a partial uniqueness condition for $\tilde{\mathbf{C}}$ by interchanging the roles of $\tilde{\mathbf{B}}\Phi$ and $\tilde{\mathbf{C}}\Omega$. Analogously, partial uniqueness conditions can be obtained if not $\tilde{\mathbf{A}}$ but $\tilde{\mathbf{B}}$ or $\tilde{\mathbf{C}}$ is unique up to permutation and scaling.

The condition on the permutation matrix Π_b in Theorem 5.4 can be relaxed to “there exists a permutation Π_b such that $\tilde{\mathbf{B}}\Pi_b = [\tilde{\mathbf{B}}_1\mathbf{S}_1|\dots|\tilde{\mathbf{B}}_F\mathbf{S}_F]$, with \mathbf{S}_f nonsingular, $f = 1, \dots, F$ ”. Since this is a more complicated condition, we chose a simpler formulation of Theorem 5.4.

6. Examples of partial uniqueness in PARALIND. Here, we demonstrate the improvement when using Theorem 5.3 instead of Theorem 5.2. Also, we show that whether Theorem 5.3 or Theorem 5.4 is more relaxed depends on the particular PARALIND decomposition. First, we continue Example 3 and show that partial uniqueness of $\tilde{\mathbf{B}}$ with respect to $\Pi_b = \mathbf{I}_6$ follows from Theorem 5.4 but not from Theorems 5.2 and 5.3. In Example 4, we construct a PARALIND decomposition such that all three matrices $\tilde{\mathbf{A}}, \tilde{\mathbf{B}}, \tilde{\mathbf{C}}$ are not unique up to permutation and scaling. To the best of our knowledge, this is the first such case in the literature. For this example, we show that partial uniqueness of $\tilde{\mathbf{B}}$ with respect to $\Pi_b = \mathbf{I}_4$ follows from Theorem 5.3 but not from Theorem 5.2. Since none of $\tilde{\mathbf{A}}, \tilde{\mathbf{B}}, \tilde{\mathbf{C}}$ are unique, Theorem 5.4 cannot be used. In our examples, we assume that $\tilde{\mathbf{A}}, \tilde{\mathbf{B}}, \tilde{\mathbf{C}}$ have full column rank.

Example 3 (continued). We have $R_1 = 3, R_2 = 6, R_3 = 4, R = 6, \Phi = \mathbf{I}_6$, and Ψ and Ω given by (4.9). In Stegeman and de Almeida [41, section 5] it was verified that $\tilde{\mathbf{A}}$ is unique by condition (2.8). Here, we show partial uniqueness of $\tilde{\mathbf{B}} = [\tilde{\mathbf{b}}_1 \tilde{\mathbf{b}}_2|\tilde{\mathbf{b}}_3 \tilde{\mathbf{b}}_4|\tilde{\mathbf{b}}_5 \tilde{\mathbf{b}}_6]$. First, we try Theorem 5.3. We set $\Pi_b = \mathbf{I}_6$. It can be verified that $\mathbf{G}_1, \mathbf{G}_2, \mathbf{G}_3$ have full column rank. We focus on the matrix

$$(6.1) \quad \Omega \operatorname{diag}(\Phi^T \mathbf{d}_f) \Psi^T = \begin{bmatrix} \alpha_1 - \alpha_2 & 0 & 0 \\ 0 & \alpha_3 - \alpha_4 & 0 \\ 0 & 0 & \alpha_5 - \alpha_6 \\ \alpha_2 & \alpha_4 & \alpha_6 \end{bmatrix},$$

with $\mathbf{d}_f = (\alpha_1 \alpha_2|\alpha_3 \alpha_4|\alpha_5 \alpha_6)^T$. It can be seen that the ranks (5.1) are $M_1^{(2)} = M_2^{(2)} = M_3^{(2)} = 1$. Any three of the following four vectors satisfy condition (5.3): $(1|0|0|0|0|0)^T, (0|0|1|0|0|0)^T, (0|0|0|0|1|0)^T, (1|1|1|1|1|1)^T$. Hence, Theorem 5.3 cannot be used to show partial uniqueness of $\tilde{\mathbf{B}}$ with respect to $\Pi_b = \mathbf{I}_6$ (and neither can Theorem 5.2).

Next, we use the fact that $\tilde{\mathbf{A}}$ is unique up to permutation and scaling, and try Theorem 5.4. We set $\Pi_a = \Pi_b = \mathbf{I}_6$, and partition $\tilde{\mathbf{A}}\Psi$ as $[\tilde{\mathbf{a}}_1 \tilde{\mathbf{a}}_1|\tilde{\mathbf{a}}_2 \tilde{\mathbf{a}}_2|\tilde{\mathbf{a}}_3 \tilde{\mathbf{a}}_3]$, and $\tilde{\mathbf{C}}\Omega$ as $[\tilde{\mathbf{c}}_1 \tilde{\mathbf{c}}_4 - \tilde{\mathbf{c}}_1|\tilde{\mathbf{c}}_2 \tilde{\mathbf{c}}_4 - \tilde{\mathbf{c}}_2|\tilde{\mathbf{c}}_3 \tilde{\mathbf{c}}_4 - \tilde{\mathbf{c}}_3]$. The three PARALIND subdecompositions are of the same form, with

$$(6.2) \quad \Psi_f = (1 \ 1), \quad \Phi_f = \mathbf{I}_2, \quad \Omega_f = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}, \quad f = 1, 2, 3.$$

Since $\tilde{\mathbf{A}}_f = [\tilde{\mathbf{a}}_f]$ and $\tilde{\mathbf{C}}_f = [\tilde{\mathbf{c}}_f \tilde{\mathbf{c}}_4]$ have full column rank, $f = 1, 2, 3$, and $(\Omega_f \odot \Psi_f)\Phi_f^T = \Omega_f$ has full column rank, $f = 1, 2, 3$, the condition of Theorem 5.4 holds and $\tilde{\mathbf{B}} = [\tilde{\mathbf{b}}_1 \tilde{\mathbf{b}}_2|\tilde{\mathbf{b}}_3 \tilde{\mathbf{b}}_4|\tilde{\mathbf{b}}_5 \tilde{\mathbf{b}}_6]$ is partially unique.

Example 4. Consider the PARALIND decomposition with $R_1 = 6$, $R_2 = R_3 = 4$, $R = 6$, $\Psi = \mathbf{I}_6$, and

$$(6.3) \quad \Phi = \begin{bmatrix} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad \Omega = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The six rank-1 terms of the decomposition are

$$(6.4) \quad \tilde{\mathbf{a}}_1 \circ \tilde{\mathbf{b}}_1 \circ \tilde{\mathbf{c}}_1 + \tilde{\mathbf{a}}_2 \circ (\tilde{\mathbf{b}}_2 - \tilde{\mathbf{b}}_1) \circ \tilde{\mathbf{c}}_1 + \tilde{\mathbf{a}}_3 \circ \tilde{\mathbf{b}}_2 \circ \tilde{\mathbf{b}}_2 + \tilde{\mathbf{a}}_4 \circ (\tilde{\mathbf{b}}_1 - \tilde{\mathbf{b}}_2) \circ \tilde{\mathbf{c}}_2 + \tilde{\mathbf{a}}_5 \circ \tilde{\mathbf{b}}_3 \circ \tilde{\mathbf{c}}_3 + \tilde{\mathbf{a}}_6 \circ \tilde{\mathbf{b}}_4 \circ \tilde{\mathbf{c}}_4.$$

The vectors in the last two rank-1 terms do not occur in the first four rank-1 terms. An alternative for the first four rank-1 terms is as follows:

$$(6.5) \quad (2\tilde{\mathbf{a}}_1 - \tilde{\mathbf{a}}_2 + \tilde{\mathbf{a}}_3 + \tilde{\mathbf{a}}_4) \circ (\tilde{\mathbf{b}}_1 - \tilde{\mathbf{b}}_2) \circ \tilde{\mathbf{c}}_1 + (\tilde{\mathbf{a}}_1 + \tilde{\mathbf{a}}_3) \circ (\tilde{\mathbf{b}}_2 - (\tilde{\mathbf{b}}_1 - \tilde{\mathbf{b}}_2)) \circ \tilde{\mathbf{c}}_1 \\ + (\tilde{\mathbf{a}}_3 + \tilde{\mathbf{a}}_4) \circ \tilde{\mathbf{b}}_2 \circ (\tilde{\mathbf{c}}_2 - \tilde{\mathbf{c}}_1) + \tilde{\mathbf{a}}_4 \circ ((\tilde{\mathbf{b}}_1 - \tilde{\mathbf{b}}_2) - \tilde{\mathbf{b}}_2) \circ (\tilde{\mathbf{c}}_2 - \tilde{\mathbf{c}}_1).$$

Hence, none of $\tilde{\mathbf{A}}, \tilde{\mathbf{B}}, \tilde{\mathbf{C}}$ is unique up to permutation and scaling. As a consequence, we cannot use Theorem 5.4 to obtain partial uniqueness results.

Next, we show partial uniqueness of $\tilde{\mathbf{B}} = [\tilde{\mathbf{b}}_1 \ \tilde{\mathbf{b}}_2 | \tilde{\mathbf{b}}_3 | \tilde{\mathbf{b}}_4]$ using Theorem 5.3. We set $\Pi_b = \mathbf{I}_4$. It can be verified that $\mathbf{G}_1, \mathbf{G}_2, \mathbf{G}_3$ have full column rank. We focus on the matrix

$$(6.6) \quad \Omega \operatorname{diag}(\Phi^T \mathbf{d}_f) \Psi^T = \begin{bmatrix} \alpha_1 & \alpha_2 - \alpha_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha_2 & \alpha_1 - \alpha_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha_4 \end{bmatrix},$$

with $\mathbf{d}_f = (\alpha_1 \ \alpha_2 | \alpha_3 | \alpha_4)^T$. It can be seen that the ranks (5.1) are $M_1^{(2)} = 2$ and $M_2^{(2)} = M_3^{(2)} = 1$. If the matrix in (6.6) should have rank 1, it follows that $\alpha_1 = \alpha_2 = 0$ and only one of α_3 and α_4 is nonzero. Hence, in condition (5.3), we obtain $\mathbf{d}_2 = (0 \ 0 | * 0)^T$ and $\mathbf{d}_3 = (0 \ 0 | 0 | *)^T$, with corresponding $\mathbf{d}_2^{(c)} = (0 \ 1 \ 0)^T$ and $\mathbf{d}_3^{(c)} = (0 \ 0 \ 1)^T$. If the matrix in (6.6) should have rank 2, it follows that either $\alpha_1 = \alpha_2 = 0$ and $\alpha_3 \alpha_4 \neq 0$, or $\alpha_1 \neq 0$ or $\alpha_2 \neq 0$ and $\alpha_3 = \alpha_4 = 0$. Since we need three linearly independent vectors $\mathbf{d}_f^{(c)}$ in condition (5.3), we obtain $\mathbf{d}_1 = (* | * 0 | 0)^T$ with $\mathbf{d}_1^{(c)} = (1 \ 0 \ 0)^T$ (where at most one of α_1 and α_2 is zero). We have $w'(\mathbf{d}_f) = 1$ for $f = 1, 2, 3$, and condition (5.3) holds. Note that the matrix in (6.6) has rank 2 for $\mathbf{d}_f = (0 \ 0 | 1 | 1)^T$, which shows that condition (5.2) of Theorem 5.2 does not hold.

Analogously, Theorem 5.3 (and not Theorem 5.2) can be used to show partial uniqueness of $\tilde{\mathbf{C}} = [\tilde{\mathbf{c}}_1 \ \tilde{\mathbf{c}}_2 | \tilde{\mathbf{c}}_3 | \tilde{\mathbf{c}}_4]$.

7. Discussion. We have proven improved and more precise variants of the main PARALIND uniqueness conditions of Stegeman and de Almeida [41]. Our proofs of the conditions of [41] as well as the new conditions are simpler than the proofs in [41], since we do not use Kruskal's [25] permutation lemma (in case of uniqueness up to permutation and scaling) or De Lathauwer's [10] equivalence lemma for partitioned matrices (in case of partial uniqueness). Our condition for uniqueness up to permutation and scaling in Theorem 2.3 is relatively easy to check by using (symbolic) linear algebra computation software. The advantage of our uniqueness conditions over those

of [41] is clearly demonstrated by means of examples from the literature. Also, we have compared our PARALIND uniqueness conditions to those obtained from the CP uniqueness conditions of Guo et al. [15]. In the case of uniqueness up to permutation and scaling, it was verified in [15] that Theorem 2.5 of [15] (for uniqueness of $\tilde{\mathbf{A}}$) is more relaxed than Theorem 2.2 of [41] when k_{Φ} and k_{Ω} are high. This is also true for our Theorem 2.3. An example where Theorem 2.5 holds but not Theorem 2.3 is [15, example 2]. On the other hand, our Theorem 2.3 is more relaxed when k_{Φ} and k_{Ω} are low. In the case of partial uniqueness, the examples in section 6 show that whether our Theorem 5.3 or Theorem 5.4 based on [15] is more relaxed depends on the particular PARALIND decomposition.

As stated in section 1, our results are proven for real-valued PARALIND decompositions. However, the proofs in Appendices A and B can be translated easily to the complex case, i.e., with complex-valued component matrices and/or complex-valued constraint matrices. To do this, we must keep in mind that our vectors live in a complex vector space \mathbb{C}^m , with inner product $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{y}^H \mathbf{x}$ and norm $\|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$, where H denotes the Hermitian or conjugated transpose. As in \mathbb{R}^m , vectors \mathbf{x} and \mathbf{y} are orthogonal when $\langle \mathbf{x}, \mathbf{y} \rangle = 0$. Also, vectors $\mathbf{x}_1, \dots, \mathbf{x}_q \in \mathbb{C}^m$ are linearly independent when $a_1 \mathbf{x}_1 + \dots + a_q \mathbf{x}_q = \mathbf{0}$ implies $a_1 = \dots = a_q = 0$ for scalars $a_1, \dots, a_q \in \mathbb{C}$. Moreover, the determinant of a complex matrix is defined to be identical to the determinant of a real matrix, and its relation to the matrix rank is identical. The considerations above imply that, in order to translate our uniqueness proofs to the complex case, we must replace the ordinary transpose T by H where orthogonality is involved. However, in cases where the transpose is due to the formulation of the decomposition such as in (A.1), the transpose should not be changed. Theorem 2.5 is proven by [15] for the complex case, and the proof of Theorem 5.4 can also be translated to the complex case.

Appendix A: Proofs of Theorems 2.2 and 2.3. Here we prove Theorem 2.3 and compare its proof to the proof of Theorem 2.2. The latter proof is different from [41], since we do not use Kruskal's [25] permutation lemma here. From the comparison of the proofs it will be clear that Theorem 2.3 is a refinement of Theorem 2.2.

First, suppose the assumptions of Theorem 2.2 hold. Since $\tilde{\mathbf{A}}$ has full column rank, [41, Lemma 3.4] implies that without loss of generality we may set $\tilde{\mathbf{A}} = \mathbf{I}_{R_1}$. We denote an alternative decomposition by $(\tilde{\mathbf{B}}\Psi, \tilde{\mathbf{B}}\Phi, \tilde{\mathbf{C}}\Omega)$, where $\tilde{\mathbf{A}}$ is an $R_1 \times R_1$ matrix. Equating the mode-1 matrix unfolding of the PARALIND decomposition (see (2.2)) to its alternative yields

$$(A.1) \quad (\tilde{\mathbf{B}} \otimes \tilde{\mathbf{C}}) \mathbf{G}_1 = (\bar{\mathbf{B}} \otimes \bar{\mathbf{C}}) \mathbf{G}_1 \tilde{\mathbf{A}}^T.$$

Since $\tilde{\mathbf{B}}$, $\tilde{\mathbf{C}}$, and \mathbf{G}_1 have full column rank by assumption, it follows that $(\tilde{\mathbf{B}} \otimes \tilde{\mathbf{C}}) \mathbf{G}_1$ has full column rank. From (A.1) it follows that also $(\bar{\mathbf{B}} \otimes \bar{\mathbf{C}}) \mathbf{G}_1$ has full column rank, and that $\tilde{\mathbf{A}}$ is nonsingular. We rewrite (A.1) as

$$(A.2) \quad (\tilde{\mathbf{B}} \otimes \tilde{\mathbf{C}}) \mathbf{G}_1 \tilde{\mathbf{A}}^{-T} = (\bar{\mathbf{B}} \otimes \bar{\mathbf{C}}) \mathbf{G}_1.$$

We denote the R_1 columns of $\tilde{\mathbf{A}}^{-T}$ as $\mathbf{d}_1, \dots, \mathbf{d}_{R_1}$. For column j of (A.2) we have

$$(A.3) \quad (\tilde{\mathbf{B}}\Phi \odot \tilde{\mathbf{C}}\Omega) \Psi^T \mathbf{d}_j = (\bar{\mathbf{B}}\Phi \odot \bar{\mathbf{C}}\Omega) \psi_j,$$

with ψ_j^T denoting row j of Ψ . We should pick linearly independent $\mathbf{d}_1, \dots, \mathbf{d}_{R_1}$ such that (A.3) holds. If it follows that $\tilde{\mathbf{A}}^{-T} = [\mathbf{d}_1 \dots \mathbf{d}_{R_1}] = \mathbf{\Pi} \mathbf{\Lambda}$, then $\tilde{\mathbf{A}} = \mathbf{\Pi} \mathbf{\Lambda}^{-1}$. Hence, in that case $\tilde{\mathbf{A}} = \mathbf{I}_{R_1}$ would be unique up to permutation and scaling.

Rewriting (A.3) in $J \times K$ matrix form yields

$$(A.4) \quad \tilde{\mathbf{B}}\Phi \operatorname{diag}(\Psi^T \mathbf{d}_j) \Omega^T \tilde{\mathbf{C}}^T = \bar{\mathbf{B}}\Phi \operatorname{diag}(\psi_j^T) \Omega^T \bar{\mathbf{C}}^T.$$

Conditions (2.8) and (2.9) are obtained by taking the rank on both sides of (A.4). Note that when taking the rank of the left-hand side of (A.4), matrices $\tilde{\mathbf{B}}$ and $\tilde{\mathbf{C}}$ can be left out since they have full column rank by assumption. In case of condition (2.8) we obtain

$$(A.5) \quad \begin{aligned} \operatorname{rank}(\Phi \operatorname{diag}(\Psi^T \mathbf{d}_j) \Omega^T) &= \operatorname{rank}(\bar{\mathbf{B}}\Phi \operatorname{diag}(\psi_j^T) \Omega^T \bar{\mathbf{C}}^T) \\ &\leq \operatorname{rank}(\Phi \operatorname{diag}(\psi_j^T) \Omega^T) \\ &= N_j^{(1)} \\ [1mm] &\leq \max(N_1^{(1)}, \dots, N_{R_1}^{(1)}). \end{aligned}$$

If (A.5) implies $w(\mathbf{d}_j) = 1$, then the linearly independent $\mathbf{d}_1, \dots, \mathbf{d}_{R_1}$ are necessarily permuted and rescaled versions of the columns of $\tilde{\mathbf{A}} = \mathbf{I}_{R_1}$. This completes the proof of Theorem 2.2.

To prove Theorem 2.3, we proceed as follows. Since $\mathbf{G}_2, \mathbf{G}_3$ and $\tilde{\mathbf{A}}, \tilde{\mathbf{B}}, \tilde{\mathbf{C}}$ have full column rank, it follows from Proposition 2.4 that also $\bar{\mathbf{B}}$ and $\bar{\mathbf{C}}$ have full column rank. Hence, we can delete $\bar{\mathbf{B}}, \bar{\mathbf{C}}$ from (A.5). The rank equality in condition (2.9) is obtained as

$$(A.6) \quad \operatorname{rank}(\Phi \operatorname{diag}(\Psi^T \mathbf{d}_j) \Omega^T) = \operatorname{rank}(\Phi \operatorname{diag}(\psi_j^T) \Omega^T) = N_j^{(1)}.$$

This completes the proof of Theorem 2.3. If the conditions of Theorem 2.3 hold, then condition (2.9) is a relaxation of condition (2.8) because the inequality $N_j^{(1)} \leq \max(N_1^{(1)}, \dots, N_{R_1}^{(1)})$ is left out in the former.

Appendix B: Proofs of Theorems 5.2 and 5.3. We proceed analogous to the proofs of Theorems 2.2 and 2.3 in Appendix A. Our proof of Theorem 5.3 is different from [41], since we do not use the equivalence lemma for partitioned matrices of De Lathauwer [10] here.

First, suppose the assumptions of Theorem 5.2 hold. We consider the PARALIND decomposition $(\tilde{\mathbf{A}}\Psi, (\tilde{\mathbf{B}}\Pi_b)(\Pi_b^T \Phi), \tilde{\mathbf{C}}\Omega)$, and denote an alternative by $(\bar{\mathbf{A}}\Psi, (\bar{\mathbf{B}}\Pi_b) \cdot (\Pi_b^T \Phi), \bar{\mathbf{C}}\Omega)$. Analogous to Appendix A, we set $\tilde{\mathbf{B}}\Pi_b = \mathbf{I}_{R_2}$ without loss of generality, and we equate the mode-2 matrix unfolding of the PARALIND decomposition (see (2.3)) to its alternative:

$$(B.1) \quad (\tilde{\mathbf{C}}\Omega \odot \tilde{\mathbf{A}}\Psi) \Phi^T \Pi_b = (\bar{\mathbf{C}}\Omega \odot \bar{\mathbf{A}}\Psi) \Phi^T \Pi_b \Pi_b^T \bar{\mathbf{B}}^T.$$

As in Appendix A, it follows that $\bar{\mathbf{B}}$ is nonsingular, and that $(\bar{\mathbf{C}}\Omega \odot \bar{\mathbf{A}}\Psi) \Phi^T$ has full column rank. Let n_f denote the number of columns in part $\bar{\mathbf{B}}_f$, $f = 1, \dots, F$. Then $\sum_{f=1}^F n_f = R_2$. Let $(\bar{\mathbf{B}}\Pi_b)^{-T} = \bar{\mathbf{B}}^{-T}\Pi_b = [\hat{\mathbf{B}}_1 | \dots | \hat{\mathbf{B}}_F]$, where the columns of $\hat{\mathbf{B}}_f$ are orthogonal to all parts of $\bar{\mathbf{B}}\Pi_b$ except part f . Then $\hat{\mathbf{B}}_f$ has n_f columns, $f = 1, \dots, F$. Note that partial uniqueness of $\bar{\mathbf{B}}$ with respect to Π_b follows if column $\mathbf{x}_j^{(f)}$ of $\hat{\mathbf{B}}_f$ satisfies $w'(\mathbf{x}_j^{(f)}) = 1$ and is not orthogonal to part $\tilde{\mathbf{B}}_{\pi(f)}$ for a permutation $\pi(\cdot)$, $j = 1, \dots, n_f$, $f = 1, \dots, F$.

Let the vector $\mathbf{d}_f \in \operatorname{span}(\hat{\mathbf{B}}_f)$ be generic, i.e., $\mathbf{d}_f = \hat{\mathbf{B}}_f \mathbf{h}$ for a generic vector \mathbf{h} . Then the vector $\mathbf{g}_f = \Pi_b^T \bar{\mathbf{B}}^T \mathbf{d}_f$ has nonzero (and generic) entries only in part f , which implies $w'(\mathbf{g}_f) = w'(\Pi_b^T \bar{\mathbf{B}}^T \mathbf{d}_f) = 1$. We write

$$(B.2) \quad (\tilde{\mathbf{C}}\Omega \odot \tilde{\mathbf{A}}\Psi) \Phi^T \Pi_b \mathbf{d}_f = (\bar{\mathbf{C}}\Omega \odot \bar{\mathbf{A}}\Psi) \Phi^T \Pi_b \mathbf{g}_f.$$

Rewriting (B.2) in $K \times I$ matrix form yields

$$(B.3) \quad \tilde{\mathbf{C}}\Omega \operatorname{diag}(\Phi^T \Pi_b \mathbf{d}_f) \Psi^T \tilde{\mathbf{A}}^T = \bar{\mathbf{C}}\Omega \operatorname{diag}(\Phi^T \Pi_b \mathbf{g}_f) \Psi^T \bar{\mathbf{A}}^T.$$

Conditions (5.2) and (5.3) are obtained by taking the rank on both sides of (B.3). Note that when taking the rank of the left-hand side of (B.3), matrices $\tilde{\mathbf{A}}$ and $\tilde{\mathbf{C}}$ can be left out since they have full column rank by assumption. In case of condition (5.2) we have

$$(B.4) \quad \begin{aligned} \operatorname{rank}(\Omega \operatorname{diag}(\Phi^T \Pi_b \mathbf{d}_f) \Psi^T) &= \operatorname{rank}(\bar{\mathbf{C}}\Omega \operatorname{diag}(\Phi^T \Pi_b \mathbf{g}_f) \Psi^T \bar{\mathbf{A}}^T) \\ &\leq \operatorname{rank}(\Omega \operatorname{diag}(\Phi^T \Pi_b \mathbf{g}_f) \Psi^T) \\ &= M_f^{(2)} \\ &\leq \max(M_1^{(2)}, \dots, M_F^{(2)}). \end{aligned}$$

According to condition (5.2), this implies $w'(\mathbf{d}_f) = 1$. Since $\mathbf{d}_f = \hat{\mathbf{B}}_f \mathbf{h}$ for a generic vector \mathbf{h} , it follows that all columns $\mathbf{x}_j^{(f)}$ of $\hat{\mathbf{B}}_f$ satisfy $w'(\mathbf{x}_j^{(f)}) = 1$ and they are not orthogonal to the same part $\tilde{\mathbf{B}}_g$. The reasoning above holds for any part f of $\bar{\mathbf{B}}^{-T} \Pi_b$. Since the latter is nonsingular, we obtain that $\bar{\mathbf{B}}^{-T} \Pi_b = \operatorname{blockdiag}(\mathbf{S}_1^{-T}, \dots, \mathbf{S}_F^{-T}) \Pi$, where block \mathbf{S}_f ($n_f \times n_f$) is nonsingular, $f = 1, \dots, F$, and Π is a block permutation matrix. This implies $\bar{\mathbf{B}} \Pi_b = \operatorname{blockdiag}(\mathbf{S}_1, \dots, \mathbf{S}_F) \Pi$, which shows that $\bar{\mathbf{B}}$ is partially unique with respect to Π_b as in Definition 5.1. This completes the proof of Theorem 5.2.

To prove Theorem 5.3, we proceed as follows. Since $\mathbf{G}_1, \mathbf{G}_3$ and $\tilde{\mathbf{A}}, \tilde{\mathbf{B}}, \tilde{\mathbf{C}}$ have full column rank, it follows from Proposition 2.4 that also $\bar{\mathbf{A}}$ and $\bar{\mathbf{C}}$ have full column rank. Hence, we can delete $\bar{\mathbf{A}}, \bar{\mathbf{C}}$ from (B.4). The rank equality in condition (5.3) is obtained as

$$(B.5) \quad \operatorname{rank}(\Omega \operatorname{diag}(\Phi^T \Pi_b \mathbf{d}_f) \Psi^T) = \operatorname{rank}(\Omega \operatorname{diag}(\Phi^T \Pi_b \mathbf{g}_f) \Psi^T) = M_f^{(2)}.$$

Now the vector \mathbf{d}_f corresponds to the rank $M_f^{(2)}$. Hence, we may have a different condition for different f . If this implies $w'(\mathbf{d}_f) = 1$ for each f , then we have partial uniqueness of $\bar{\mathbf{B}}$ with respect to Π_b as explained above. The linear independence of the vectors $\mathbf{d}_1^{(c)}, \dots, \mathbf{d}_F^{(c)}$ corresponds to the nonsingularity of $\bar{\mathbf{B}}^{-T} \Pi_b = [\hat{\mathbf{B}}_1 | \dots | \hat{\mathbf{B}}_F]$. This completes the proof of Theorem 5.3. If the conditions of Theorem 5.3 hold, then condition (5.3) is a relaxation of condition (5.2) because the inequality $M_f^{(2)} \leq \max(M_1^{(2)}, \dots, M_F^{(2)})$ is left out in the former.

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